A Third Order Nakashima Pseudo-Runge-Kutta Method

LIM TIAN HWEE
Foundation in Business,
University of Reading Malaysia,
Level 7, Menara Kotaraya, Jalan Trus, Johor Bahru, Johor Darul Ta'zim
E-mail: t.h.lim@reading.ac.uk

ABSTRACT

In this paper, we present a third order Nakashima pseudo-Runge-Kutta method. This method is derived by minimizing the error bound to determine the free parameters. Since the proposed method is only two-stages, it is cheaper than the traditional method. The stability of the method is analyzed and the numerical results are tabulated to compare with the traditional method.

Keywords: Nakashima pseudo-Runge-Kutta method, third order, initial value problem.

INTRODUCTION

Explicit Runge-Kutta method is probably the most popular method for initial value problems. An *s*-stage explicit Runge-Kutta (ERK) method may be written in the form

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i K_i$$
$$K_i = f(t_n + c_i h, y_n + h\sum_{j=0}^{i-1} a_{ij} K_j) \quad (1.1)$$

If $P(s)$ denotes the order that can be attained by an s-stage ERK method (1.1), Butcher [2] proved

$$P(s) = s \text{ for } s = 1, 2, 3, 4 \quad (1.2)$$

As we can see, the *s*-stage ERK method (1.1), from order one to order four requires exactly *s* functional evaluations for step.

Bryne [1] is the first person to introduce the pseudo Runge-Kutta method. The method he proposed requires fewer functional evaluations than (1.1). However, Bryne's method [1] is less accurate than (1.1). Nakashima [6] introduced a type of pseudo-Runge-Kutta method which is cheaper and more efficient in terms of accuracy than Bryne's method.

Based on Nakashima's idea, we construct a third order pseudo-Runge-Kutta method to solve initial value problems. The method proposed requires only two stages, thus it is still computationally cheaper than ERK (1.1)

Derivation of the Method

Nakashima pseudo-Runge-Kutta method (PRK) [6] can be written as follows

$$y_{n+1} = y_n + h\sum_{i=0}^{s} b_i K_i$$

$$K_i = f(t_n + c_i h, y_i + \lambda_i (y_n - y_{n-1}) + h\sum_{j=0}^{i-1} a_{ij} K_j)$$

$$c_i = \lambda_i + \sum_{j=0}^{i-1} a_{ij} \qquad i = 2,\cdots,s$$

(2.1)

for $c_0 = \lambda_0 = -1$, $c_1 = \lambda_1 = 0$ and $0 \le c_s \le 1$. In [6], Nakashima proved that the pseudo-Runge-Kutta method (1.3) has the order

$P(s) = s + 2$ for $s =$ 2, 3, 4

(2.2)

We found from Nakashima [6] and Shintani [8], the following eight order conditions, which are required to construct a third order PRK method.

1.	$\tau_1^{(1)} : \sum_i b_i = 1$
2.	$\tau_1^{(2)} : \sum_i b_i c_i = \frac{1}{2}$
3.	$\tau_1^{(3)} : \sum_i b_i c_i^2 = \frac{1}{3}$
4.	$\tau_2^{(3)} : -\sum_i b_i \lambda_i + 2\sum_{ij} b_i a_{ij} c_j = \frac{1}{3}$

Table 2.1: Third order pseudo-Runge-Kutta order conditions

From Table 2.1, there are only four equations to be satisfied and we have six unknowns, thus we have two arbitrary parameters to determine. After solving all the related equations, we have

$$b_0 = -\frac{1}{6}\frac{3c_2 - 2}{c_2 + 1}, \quad b_1 = \frac{1}{6}\frac{9c_2 - 5}{c_2}, \quad b_2 = \frac{5}{6}\frac{1}{c_2(c_2 + 1)}$$

$$\lambda_2 = -c_2^2 - 2a_{20} \tag{2.4}$$

where c_2 and a_{20} are free parameters which we want to determine.

5.	$\tau_1^{(4)} : \sum_i b_i c_i^3 = \frac{1}{4}$
6.	$\tau_2^{(4)} : -\sum_i b_i c_i \lambda_i + 2\sum_{ij} b_i c_i a_{ij} c_j = \frac{1}{4}$
7.	$\tau_3^{(4)} : \sum_i b_i \lambda_i + 3\sum_{ij} b_i a_{ij} c_j^2 = \frac{1}{4}$
8.	$\tau_4^{(4)} : -\sum_{ij} b_i c_j^2 a_{ij} - \sum_i b_i \lambda_j + 6\sum_{ij} b_i a_{ij} a_{jk} c_k = 0$

Table 2.2: Error factors for third order pseudo-Runge-Kutta method

All four error factors will become zero for $c_2 = 0.7$ and $a_{20} = 0.833$. However, this also means for this choice of c_2 and a_{20}, our third order method would actually become a fourth order method. However, by choosing for c_2 a value close to $\frac{7}{10}$, namely $c_2 = \frac{5}{7}$, we expect that the error factor will become small.

We use the notation from Lotkin [5] and Ralston [7] to determine the error bounds on E for our third order PRK method.

$$|E| \le CML^3h^4 \tag{2.5}$$

where C is the error constant in a region $\mathbb{R}$ about (t_n, y_n)

$$|f(x,y)| < M \quad \text{and} \quad \left|\frac{f^{i+j}(x,y)}{\partial x^i \partial y^j}\right| < \frac{L^{i+j}}{M^{j-1}} \tag{2.6}$$

where L and M are positive constants independent of t, y. With $c_2 = \frac{5}{7}$, the constant C is estimated by

$$C = \left|\frac{5}{126}\right| + \left|-\frac{7}{18} + \frac{49a_{20}}{108}\right| + \left|-\frac{7}{36} + \frac{49a_{20}}{216}\right| \quad (2.7)$$

Our objective is to minimize the right-hand side of (2.7). We found that the bound of C is minimized when $a_{20} = \frac{6}{7}$.

Substituting the value c_2 and a_{20} into (2.4), we obtain a two-stage third order pseudo-Runge-Kutta method

$$y(t_i + h) = y_i + \frac{h}{72}(-k_0 + 24k_1 + 49k_2)$$

where

$$\begin{aligned}
k_0 &= f(t_{i-1}, y_{i-1}) \\
k_1 &= f(t_i, y_i) \\
k_2 &= f\left(t_{i-1} + \frac{5}{7}h, y_i - \frac{109}{49}(y_i - y_{i-1}) + \frac{6}{7}hk_0 + \frac{102}{49}hk_1\right)
\end{aligned} \quad (2.8)$$

The local truncation error for formula (2.8) satisfies

$$|E| \le \frac{5}{126} ML^3h^4 \quad (2.9)$$

We present our new third order PRK method in the tableau

$$\begin{array}{c|c|ccc}
-1 & -1 & & & \\
0 & 0 & & & \\
\frac{5}{7} & -\frac{109}{49} & \frac{6}{7} & \frac{102}{49} & \frac{56}{193} \\
\hline
 & & -\frac{1}{72} & \frac{24}{72} & \frac{49}{72}
\end{array}$$

Ralston [7] presents a third order Runge-Kutta method

0			
$\frac{1}{2}$	$\frac{1}{2}$		
$\frac{3}{4}$	0	$\frac{3}{4}$	
	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$

We will compare the effectiveness of the proposed method with Ralston Runge-Kutta method. Notice that the new proposed third order method requires only two stages. It is cheaper than the traditional third order explicit Runge-Kutta method.

Stability Analysis

To determine the stability function of the proposed method, we applied the famous Dahlquist's test equation

$$y' = f(x,y) = \lambda\, y \tag{3.1}$$

to formula (2.8). The stability polynomial for the proposed pseudo-Runge-Kutta method is

$$y_{i+1} = y_i + \frac{3}{2}h\lambda\, y_{i-1} - \frac{1}{2}h\lambda\, y_i + \frac{7}{12}h^2\lambda^2 y_{i-1} + \frac{7}{12}h^2\lambda^2 y_i \tag{3.2}$$

On assuming $h\lambda = z$, $y_{i+1} = \zeta$, $y_i = \zeta^0$ and $y_{i-1} = \zeta^{-1}$ equation (3.2) becomes

$$\zeta - 1 + \frac{3z}{2\zeta} + \frac{z}{2} - \frac{7z^2}{12\zeta} - \frac{17z^2}{12} = 0 \tag{3.3}$$

Solving equation (3.3) we have two roots

$$\zeta_1 = \frac{1}{2} - \frac{1}{4}z + \frac{17}{24}z^2 + \frac{\sqrt{144 + 720z + 780z^2 - 204z^3 + 289z^4}}{24}$$
$$\zeta_2 = \frac{1}{2} - \frac{1}{4}z + \frac{17}{24}z^2 - \frac{\sqrt{144 + 720z + 780z^2 - 204z^3 + 289z^4}}{24} \tag{3.4}$$

According to Yaacob et. al [10], $|\zeta_1| \leq 1$ and $|\zeta_2| \leq 1$ are two stability functions for the new PRK method. By taking $z = x + yi$, we plot the stability region using MAPLE package. The shaded region is the region which satisfies the condition $|\zeta_1| \leq 1$ and $|\zeta_2| \leq 1$. The stability region for the new method is given in Figure 3.1.

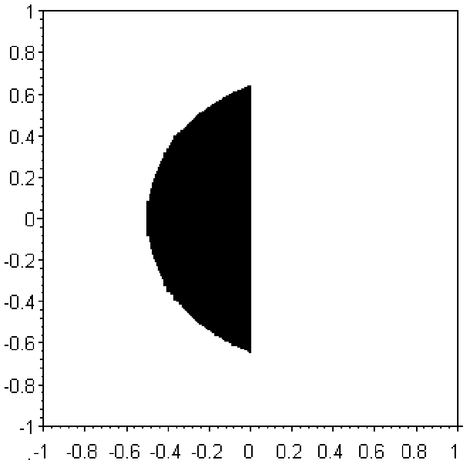

Figure 3.1: Stability Region the third order PRK method (2.8)

Numerical Results

We solved the following initial value problems using formula (2.9) and Ralston third order ERK method [7]. The necessary value y_1 for the proposed PRK method (2.9) is computed by Ralston third order ERK method.

Problem 1: The negative exponential

$$y'(t) = -y, \quad y(0) = 1$$

Exact solution: $y(t) = e^{-t}$

Results are given for $t \in [0,1]$

Problem 2: Special case for Raccati equation

$$y'(t) = -\frac{y^3}{2}, \quad y(0) = 1$$

Exact solution: $y(t) = \dfrac{1}{\sqrt{1+t}}$

Results are given for $t \in [0,1]$

Problem 3: Logistic curve

$$y'(t) = \frac{y}{4}\left(1 - \frac{y}{20}\right), \quad y(0) = 1$$

Exact solution: $y(t) = \dfrac{20}{1 + 19e^{-t/4}}$

Results are given for $t \in [0,1]$

Below are the notations used:

Ralston RK3: Ralston Third order ERK method
Pseudo RK3 : Third order PRK method (2.9)
MAXERR : Maximum error $| y(x_i) - y_i |$
H : Step size used

1.6607 (-5) means 1.6607 $\times 10^{-5}$

No	Method	H	MAXERR
1	Ralston RK3	0.1	1.6607 (-5)
	Pseudo RK3		4.0847 (-6)
2	Ralston PRK3	0.05	1.9943 (-6)
	Pseudo RK3		2.5783 (-7)
3	Ralston RK3	0.01	1.5451 (-8)
	Pseudo RK3		4.1584 (-10)
4	Ralston RK3	0.005	1.9237 (-9)
	Pseudo RK3		2.6015 (-11)
5	Ralston RK3	0.001	1.5331 (-11)
	Pseudo RK3		4.1659 (-14)

Table 4.1: Maximum absolute errors for Problem 1

No	Method	H	MAXERR
1	Ralston RK3	0.1	1.1975 (-5)
	Pseudo RK3		6.0350 (-6)
2	Ralston PRK3	0.05	1.4241 (-6)
	Pseudo RK3		4.1013 (-7)
3	Ralston RK3	0.01	1.0949 (-8)
	Pseudo RK3		1.3476 (-9)
4	Ralston RK3	0.005	1.3617 (-9)
	Pseudo RK3		1.5437 (-10)
5	Ralston RK3	0.001	1.0856 (-11)
	Pseudo RK3		1.1474 (-12)

Table 4.2: Maximum absolute errors for Problem 2

No	Method	H	MAXERR
1	Ralston RK3	0.1	1.3247 (-7)
	Pseudo RK3		1.6690 (-8)
2	Ralston PRK3	0.05	1.6705 (-8)
	Pseudo RK3		1.2327 (-9)
3	Ralston RK3	0.01	1.3458 (-10)
	Pseudo RK3		4.0905 (-12)
4	Ralston RK3	0.005	1.6837 (-11)
	Pseudo RK3		4.1854 (-13)
5	Ralston RK3	0.001	1.3475 (-13)
	Pseudo RK3		2.7516 (-15)

Table 4.2: Maximum absolute errors for Problem 3

Problem 4: SIS Model

$S'(t) = -rSI + \alpha I, \quad S(0) = S_0$

$I'(t) = rSI - \alpha I, \quad I(0) = I_0$

Exact solution: $$\begin{cases} S(t) = N - \dfrac{\beta}{r + \left(\dfrac{\beta - rI_0}{I_0}\right)e^{-\beta t}} \\ I(t) = \dfrac{\beta}{r + \left(\dfrac{\beta - rI_0}{I_0}\right)e^{-\beta t}} \end{cases}$$

Where $\beta = rN - \alpha$

Results are given for $t \in [0,1]$

This is the famous SIS model, where N denotes the size of population, $S(t)$ and $I(t)$ both denote the susceptible population and the infective population, where $S(t) + I(t) = N$, with the initial condition $S_0 > 0$, $I_0 > 0$, and constant $r > 0$ and $\alpha > 0$. Figure 4.1 and 4.2 illustrate the numerical results give by both methods with $S_0 = 200$, $I_0 = 50$, and $r = 0.04$ and $\alpha = 0.5$.

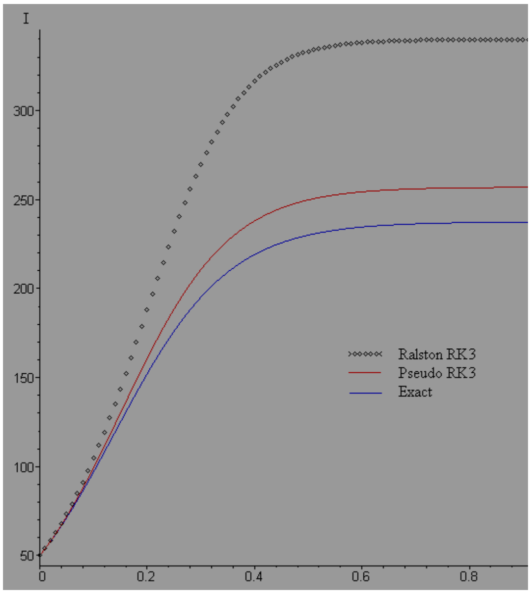

Figure 4.1: Numerical solution for $I(t)$

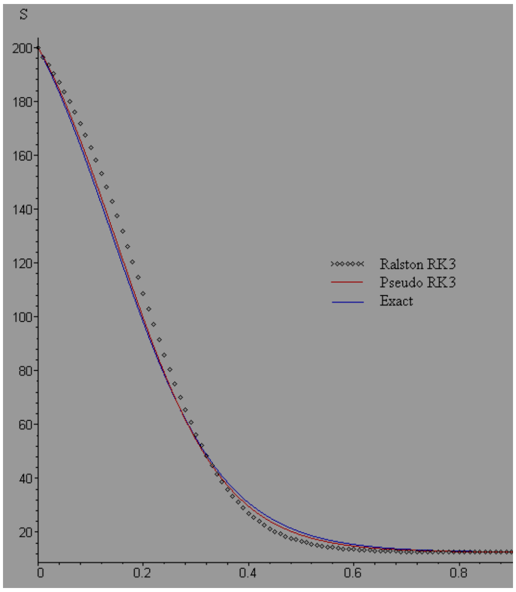

Figure 4.2: Numerical solution for $S(t)$

CONCLUSION

We derived a third order Nakashima pseudo-Runge-Kutta method based on Nakashima's idea [6]. The proposed method is then compared with a three-stage third order Runge-Kutta method proposed by Ralston in terms of efficiency in solving initial value problems. The numerical results show the presented method gives more accurate solutions compared with the Ralston third order ERK method. Since this method has only two stages, it is computationally cheaper than the other traditional third order ERK method.

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