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Chromatic Equivalence Classes of Some Families of Complete Tripartite Graphs

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Abstract. We obtain new necessary conditions on a graph which shares the same chromatic polynomial as that of the complete tripartite graph $K_{m,n,r}$. Using these, we establish the chromatic equivalence classes for $K_{1,n,n+1}$ (where $n \geq 2$). This gives a partial solution to a question raised earlier by the authors. With the same technique, we further show that $K_{n-3,n,n+1}$ is chromatically unique if $n \geq 5$. In the more general situation, we show that if $2 \leq m \leq n$, then $K_{m,n,n+1}$ is chromatically unique if n is sufficiently large.

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1. Introduction

All graphs mentioned in this paper are finite, undirected having neither loops nor multiple edges. Let G be a graph and let $P(G;\lambda)$ denote its chromatic polynomial. The *chromatic equivalence class* of G, denoted $\mathscr{C}(G)$, is the set of all graphs sharing the same chromatic polynomial with that of G. In the event that $\mathscr{C}(G) = \{G\}$, then G is said to be *chromatically unique*. The search for chromatic equivalence classes of graphs has been the subject of much interest in chromatic graph theory (see [5] for a review on the topic).

In what follows, we let K_n denote a complete graph on n vertices. Suppose G and H are two graphs. Let $G+H$ denote the graph obtained by joining every vertex of G to every vertex of H. Suppose $K_{m,n,r}$ denotes the complete tripartite graph whose partite sets have cardinalities m, n and r. Then clearly $K_{m,n,r} = \overline{K}_m + \overline{K}_n + \overline{K}_r$ where $\overline{G}$ denotes the complement of the graph G. Note that the chromaticity of $K_m + G$ has also been studied earlier in [1], where G denotes some chromatically unique graphs. More about the chromatic equivalence class of (join of) graphs can also been found in [2].

While the chromatic equivalence classes for the complete bipartite graphs have been completely settled (see [5]), not much is known about the chromatic equivalence class for

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the complete tripartite graphs although the problem has been studied since 1988 (see [3]). Some recent results on the chromaticity of complete multi-partite graphs can be found in [8]. In this paper, we focus our attention on finding the chromatic equivalence class for the complete tripartite graphs of the type $K_{m,n,n+1}$. For this purpose, some necessary conditions for a graph to share the same chromatic polynomial as that of $K_{m,n,n+1}$ are developed in Section 2, the main one being Theorem 2.1.

Let $\mathscr{T}_m$ denote the set of all trees on m vertices and let $\mathscr{J}(m,n) = \{T+\overline{K}_m, S+\overline{K}_n \mid T \in \mathscr{T}_{n+1}, S \in \mathscr{T}_{m+1}\}$. Since $K_{1,m,n} = \overline{K}_1+\overline{K}_m+\overline{K}_n$, it follows readily that $K_{1,m,n}$, $T+\overline{K}_m$ and $S+\overline{K}_n$ all have the same chromatic polynomial. Hence $\mathscr{J}(m,n) \subseteq \mathscr{C}(K_{1,m,n})$. In [4] it was shown that $\mathscr{C}(K_{1,n,n}) = \mathscr{J}(n,n)$ for any positive integer n and that $\mathscr{C}(K_{1,r,4}) = \mathscr{J}(r,4)$ if $r \in \{2,3\}$. Further it was asked whether or not $\mathscr{C}(K_{1,m,n}) = \mathscr{J}(m,n)$. In the present paper, we show that $\mathscr{C}(K_{1,n,n+1}) = \mathscr{J}(n,n+1)$ (Theorem 3.1). It looks very much likely that $\mathscr{C}(K_{1,m,n}) = \mathscr{J}(m,n)$.

Conjecture 1.1. *$\mathscr{C}(K_{1,m,n}) = \mathscr{J}(m,n)$ for all positive integers $m,n \geq 2$.*

Using the same method, we move on to show that (i) $K_{n-3,n,n+1}$ is chromatically unique if $n \geq 5$ (Theorem 3.2) and that (ii) $K_{m,n,n+1}$, where $2 \leq m \leq n$, is chromatically unique if n is sufficiently large (Theorem 3.3).

2. Some necessary conditions

Let G be a graph on p vertices and q edges and let $n(A^*,G)$ denote the number of induced subgraphs in G that are isomorphic to A. A spanning subgraph is called *special* if its connected components are complete graphs. Let $s_i(G)$ denote the number of special spanning subgraphs of G with i components, $i = 1,2,\ldots,p$. Then, following Frucht [7], the chromatic polynomial of G may be expressed as

$$P(G;\lambda) = \sum_{i=1}^{p} s_i(\overline{G})(\lambda)_i$$

where $(\lambda)_i = \lambda(\lambda-1)\cdots(\lambda-i+1)$ is the falling factorial and $\overline{G}$ is the complement of G. It is clear that $s_p(\overline{G}) = 1$ and $s_{p-1}(\overline{G}) = \overline{q}$ if $\overline{G}$ has $\overline{q}$ edges.

Note that if $Y \in \mathscr{C}(G)$, then $s_i(\overline{Y}) = s_i(\overline{G})$ for all $\chi(G) \leq i \leq p$, where $\chi(G)$ is the chromatic number of G. Thus, it follows that Y and G have the same numbers of vertices and edges. Furthermore, in the event that G contains no K_4, it follows from Theorem 1 of [6] that $n(C_4^*,Y) = n(C_4^*,G)$. Here C_4 denotes a cycle with 4 vertices.

Let $\mathscr{K}^e(s_1,s_2,s_3)$ denote the set of all connected tripartite graphs obtained by deleting e edges from the complete tripartite graph K_{s_1,s_2,s_3}. Note that, for any graph $Y \in \mathscr{K}^e(s_1,s_2,s_3)$, $\overline{Y}$ is the disjoint union of three complete subgraphs K_{s_1}, K_{s_2} and K_{s_3} with e edges joining these subgraphs.

Suppose, for any triplet (i,j,k) where $\{i,j,k\} = \{1,2,3\}$, that there are a_i edges joining the subgraphs K_{s_j} and K_{s_k}. Then $a_1+a_2+a_3 = e$. Let E_i denote the set of all the a_i edges joining K_{s_j} and K_{s_k} where $i = 1,2,3$. Two edges $\alpha \in E_r$ and $\beta \in E_s$, where $r \neq s$, are said to be a *coincidence pair* of Y if they are incident with each other in $\overline{Y}$.

Suppose $Y \in \mathscr{C}(G)$. We shall now record some known necessary conditions on Y as well as develop new ones.

Lemma 2.1. [4] *Let G be the complete tripartite graph K_{m_1,m_2,m_3}.*

(i) *If* $Y \in \mathscr{C}(G)$, *then* $Y \in \mathscr{K}^e(s_1,s_2,s_3)$ *where* $e = \sum_{i<j} s_i s_j - \sum_{i<j} m_i m_j$.

(ii) *Suppose* $Y \in \mathscr{K}^e(s_1,s_2,s_3)$ *and* $s_1+s_2+s_3 = m_1+m_2+m_3$. *Then, for each* $j \in \{1,2,3\}$,

$$s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) \geq \prod_{i=1}^{3}(s_i - m_j) - \sum_{i=1}^{3} a_i(s_i - m_j)$$

and equality holds if and only if Y *has no coincidence pair.*

Corollary 2.1. *Let* G *be the complete tripartite graph* K_{m_1,m_2,m_3} *and* $Y \in \mathscr{K}^e(s_1,s_2,s_3)$ *where* $s_1 \leq s_2 \leq s_3$. *Then*

$$s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) \geq (s_3-m_1)(s_3-m_2)(s_3-m_3).$$

Proof. From Lemma 2.1(ii) with $j=1$, we have

$$\begin{aligned} s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) &\geq \prod_{i=1}^{3}(s_i - m_1) - \sum_{i=1}^{3} a_i(s_i - m_1) \\ &\geq \prod_{i=1}^{3}(s_i - m_1) - (a_1+a_2+a_3)(s_3-m_1) \\ &= (s_3-m_1)\{(s_1-m_1)(s_2-m_1)-e\} \end{aligned}$$

Since $e = s_1s_2+s_1s_3+s_2s_3-m_1m_2-m_1m_3-m_2m_3$ (by Lemma 2.1(i)) and $s_1+s_2+s_3 = m_1+m_2+m_3$, the expression $(s_1-m_1)(s_2-m_1)-e$ can readily be simplified to $(s_3-m_2)(s_3-m_3)$ and the proof is complete. ∎

Lemma 2.2. *Let* G *be the complete tripartite graph* K_{m_1,m_2,m_3} *and* $Y \in \mathscr{K}^e(s_1,s_2,s_3)$. *Suppose further that* $Y \in \mathscr{C}(G)$ *and* $1 \leq m_1 \leq m_2 \leq m_3$ *and* $e > 0$. *Then* $s_2 < m_3$ *or* $s_3 < m_3$ *if* $s_1 \leq s_2 \leq s_3$.

Proof. Let $f(x_1,x_2,x_3) = x_1x_2+x_2x_3+x_3x_1$. Then we can show that if $s_1+s_2+s_3 = m_1+m_2+m_3$ and $s_2 \geq m_2, s_3 \geq m_3$, then

$$f(s_1,s_2,s_3) \leq f(m_1,m_2,m_3)$$

where equality holds only if $s_i = m_i$ for $i = 1,2,3$. Moreover, under the condition $s_1+s_2+s_3 = m_1+m_2+m_3$ and $s_2 \geq m_2, s_3 \geq m_3$, $f(s_1,s_2,s_3)$ attains its maximum value when $s_2+s_3 = m_2+m_3$.

To see this, suppose $s_2 > m_2$ and $s_3 \geq m_3$. Then $s_1 < m_1 \leq m_2 < s_2$ implying that $s_2 - s_1 \geq 2$. Hence

$$f(s_1+1,s_2-1,s_3) = (s_1+1)(s_2-1)+s_1s_3+s_2s_3 = f(s_1,s_2,s_3)+s_2-s_1-1 > f(s_1,s_2,s_3).$$

Similarly, if $s_2 \geq m_2$ and $s_3 > m_3$, we also have

$$f(s_1+1,s_2,s_3-1) > f(s_1,s_2,s_3).$$

Hence $f(s_1,s_2,s_3) \leq f(m_1,m_2,m_3)$ whenever $s_1+s_2+s_3 = m_1+m_2+m_3$ and $s_2 \geq m_2, s_3 \geq m_3$. ∎

Theorem 2.1. *Let* G *be the complete tripartite graph* K_{m_1,m_2,m_3} *and* $Y \in \mathscr{K}^e(s_1,s_2,s_3)$ *where* $s_1 \leq s_2 \leq s_3$. *Suppose further that* $Y \in \mathscr{C}(G)$ *and* $1 \leq m_1 \leq m_2 < m_3$. *Then* $m_2 \leq s_3 \leq m_3$. *Furthermore,*

(i) *if* $s_3 = m_2$*, then either* $Y \cong H + \overline{K}_{m_2}$ *for some bipartite graph* H *or else* $Y \in \mathscr{K}^e(m_1 + m_3 - m_2, m_2, m_2)$ *where* $e = (m_3 - m_2)(m_2 - m_1)$*, and*

(ii) *if* $s_3 = m_3$*, then* $Y \cong H + \overline{K}_{m_3}$ *for some bipartite graph* H*.*

Proof. Suppose $s_3 > m_3$. Then $s_3 > m_3 \geq m_2 \geq m_1$. By Corollary 2.1, we have $s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) > 0$, a contradiction because $Y \in \mathscr{C}(G)$. Therefore $s_3 \leq m_3$.

Suppose on the contrary that $s_3 < m_2$. Then we have $s_1 \leq s_2 \leq s_3 < m_2 \leq m_3$. By Corollary 2.1, we have

$$s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) \geq (s_3 - m_1)(s_3 - m_2)(s_3 - m_3)$$

Now, if $s_3 > m_1$, then $s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) > 0$, a contradiction because $Y \in \mathscr{C}(G)$. On the other hand, if $s_3 \leq m_1$, then $s_1 \leq s_2 \leq m_1$ implies $s_1 + s_2 + s_3 \leq 3m_1 < m_1 + m_2 + m_3$ which is impossible. Therefore $s_3 \geq m_2$.

(i) Suppose $s_3 = m_2$.

If $s_2 \neq m_2$, then $s_1, s_2 < m_2$. By Lemma 2.1(ii) with $j = 2$, we have

$$s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) \geq -a_1(s_1 - m_2) - a_2(s_2 - m_2).$$

Since $Y \in \mathscr{C}(G)$, $s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = 0$ and this implies $a_1 = a_2 = 0$. Therefore $e = a_3$ and $Y \cong H + \overline{K}_{s_3} \cong H + \overline{K}_{m_2}$ for some bipartite graph H. On the other hand, if $s_2 = m_2$, then $s_1 = m_1 + m_2 + m_3 - (s_2 + s_3) = m_1 + m_3 - m_2$ and this implies $Y \in \mathscr{K}^e(m_1 + m_3 - m_2, m_2, m_2)$ where $e = (m_3 - m_2)(m_2 - m_1)$ by Lemma 2.1(i).

(ii) Suppose $s_3 = m_3$.

Then $s_1, s_2 < m_3 = s_3$ by Lemma 2.2 (because $s_1 \leq s_2 < m_3 = s_3$). By Lemma 2.1(ii) with $j = 3$, we have

$$s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) \geq -a_1(s_1 - m_3) - a_2(s_2 - m_3).$$

Since $Y \in \mathscr{C}(G)$, $s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = 0$ and hence $a_1 = a_2 = 0$. This implies that $e = a_3$ and $Y \cong H + \overline{K}_{s_3} \cong H + \overline{K}_{m_3}$ for some bipartite graph H. This completes the proof. ∎

3. Results

Recall the following result from [4].

Lemma 3.1. [4] *Let G be the complete tripartite graph* K_{m_1,m_2,m_3} *and* $Y \in \mathscr{K}^e(s_1, s_2, s_3)$*.*

(i) *Suppose further that* $Y \in \mathscr{C}(G)$*,* $2 \leq m_1 \leq m_2 \leq m_3$ *and* $Y \cong H + \overline{K}_t$ *for some bipartite graph H and some* $t \in \{m_1, m_2, m_3\}$*. Then Y is isomorphic to G.*

(ii) *Suppose further that* $Y \cong H + \overline{K}_n$ *where H is a bipartite graph and n is a positive integer. If H is disconnected, then* $s_3(\overline{Y}) > s_3(\overline{G})$*.*

We can now prove that Conjecture 1.1 is true for the complete tripartite graph $K_{1,n,n+1}$.

Theorem 3.1. *For any positive integer* $n \geq 2$*,* $\mathscr{C}(K_{1,n,n+1}) = \mathscr{J}(n, n+1)$*.*

Proof. We need only to show that $\mathscr{C}(K_{1,n,n+1}) \subseteq \mathscr{J}(n, n+1)$. Let G denote the complete tripartite graph $K_{1,n,n+1}$ and suppose $Y \in \mathscr{C}(G)$. By Theorem 2.1, either $Y \cong H_1 + \overline{K}_n$ or $Y \cong H_2 + \overline{K}_{n+1}$ for some bipartite graphs H_1 and H_2 or else $Y \in \mathscr{K}^e(2, n, n)$ where $e = n - 1$. By Lemma 3.1(ii), either of the subgraphs H_1 and H_2 is connected. Hence $H_1 \in \mathscr{T}_{n+2}$ and $H_2 \in \mathscr{T}_{n+1}$ because the numbers of edges in H_1 and H_2 are $n+1$ and n respectively. But this means that $Y \in \mathscr{J}(n,n+1)$. On the other hand, if $Y \in \mathscr{K}^e(2,n,n)$ where $e = a_1 + a_2 + a_3 = n-1$, then by Lemma 2.1(ii) with $j = 2$, we have

$$s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) \geq -a_1(2-n).$$

Since $Y \in \mathscr{C}(G)$, $s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = 0$ and this implies that either $n = 2$ or else $n > 2$ and $a_1 = 0$. If $n = 2$, then $e = 1$ and $Y \cong T + \overline{K}_2$ where T is a path on 4 vertices and hence $Y \in \mathscr{J}(2,3)$.

Therefore assume that $n > 2$ and $a_1 = 0$. Since $s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = 0$, by Lemma 2.1(ii), Y has no coincidence pair. As such, the subgraph induced by the a_2 edges in E_2 (respectively the a_3 edges in E_3) in $\overline{Y}$ is isomorphic to K_{1,a_2} (respectively K_{1,a_3}). Therefore we have,

$$\begin{aligned} n(C_4^*,Y) &= \binom{n}{2}^2 + 2\binom{n}{2} + (a_2+a_3)\binom{n}{2} - \sum_{i=2}^{3} a_i(n-a_i) - \sum_{i=2}^{3}\binom{a_i}{2} \\ &= \binom{n}{2}^2 + 2\binom{n}{2} + (a_2+a_3)(\binom{n}{2} - n) + \binom{a_2+1}{2} + \binom{a_3+1}{2} \\ &= \binom{n}{2}^2 + n\binom{n}{2} - a_2a_3 \\ &= n(C_4^*,G) - a_2a_3 \end{aligned}$$

This implies that either $a_2 = 0$ and $a_3 = e = n-1$ or else $a_3 = 0$ and $a_2 = e = n-1$. Either case implies that $Y \cong H + \overline{K}_n$ for some bipartite graph H. By Lemma 3.1(ii), H is connected. Note that H has $n+2$ vertices and $2n - e = n+1$ edges. That is, $H \in \mathscr{T}_{n+2}$ and hence $Y \in \mathscr{J}(n,n+1)$. The proof is now complete. ∎

Next, we show that $K_{n-3,n,n+1}$ is chromatically unique if $n \geq 5$. In what follows, we let $A(m,n) = \binom{n}{2}^2 + 2\binom{m+1}{2}\binom{n}{2} + (n-m)\binom{n}{2}$. Then we have $A(m,n) = n(C_4^*, K_{m,n,n+1}) + \frac{1}{2}mn(n-m)$.

Theorem 3.2. *For any integer $n \geq 5$, $K_{n-3,n,n+1}$ is chromatically unique.*

Proof. Let G denote the graph $K_{n-3,n,n+1}$. Assume that $Y \in \mathscr{C}(G)$ and Y is not isomorphic to G. Applying Theorem 2.1 and Lemma 3.1(i), it follows that $Y \in \mathscr{K}^3(n-2,n,n)$. By Lemma 2.1(ii) with $j = 2$, we have

$$s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) \geq 2a_1.$$

Since $Y \in \mathscr{C}(G)$, we must have $s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = 0$ and this implies that $a_1 = 0$. Note that $e = 3$, that is, $a_2 + a_3 = 3$. Let $E_2 \cup E_3 = \{e_1, e_2, e_3\}$. Note that neither E_2 nor E_3 is an empty set because otherwise $Y \cong H + \overline{K}_n$ for some bipartite graph H, which by Lemma 3.1(i), implies Y is isomorphic to G. Without loss of generality, we may assume that $e_1, e_2 \in E_2$ and $e_3 \in E_3$. Since $s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = 0$, by Lemma 2.1(ii), Y has no coincidence pair. Thus there are three possible cases for Y : e_1 and e_2 are not incident, or they have a common vertex in the partite set having n vertices, or a common vertex in the partite set having $n-2$ vertices. Let X_1, X_2, X_3 represent Y corresponding to these three cases.

It is routine to check that for each $i \in \{1,2,3\}$, $n(C_4^*, X_i) = A(n-3,n) - x_i$ where $x_1 = 3n^2 - 12n + 8$, $x_2 = 3n^2 - 13n + 10$ and $x_3 = 3n^2 - 13n + 12$. Since $A(n-3,n) = n(C_4^*, K_{n-3,n,n+1}) + 3n(n-3)/2$, it follows that $n(C_4^*, X_i) < n(C_4^*, G)$ for each $i \in \{1,2,3\}$ and the proof is complete. ∎

Theorem 3.3. *Suppose m and n are natural numbers such that $2 \leq m \leq n$. Then there exists a natural number $N(m)$ (depending on m) such that $K_{m,n,n+1}$ is chromatically unique whenever $n \geq N(m)$.*

Proof. Let G denote the graph $K_{m,n,n+1}$. Assume that $Y \in \mathscr{C}(G)$ and Y is not isomorphic to G. Applying Theorem 2.1 and Lemma 3.1(i), it follows that $Y \in \mathscr{K}^e(m+1,n,n)$ where $e = n-m$. We shall obtain a contradiction by showing that if n is sufficiently large, then $n(C_4^*,Y) < n(C_4^*,K_{m,n,n+1})$ for any $Y \in \mathscr{K}^e(m+1,n,n)$ where $e = n-m$. By Lemma 2.1(ii) with $j = 2$, we have

$$s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) \geq (n-(m+1))a_1.$$

Since $Y \in \mathscr{C}(G)$, we must have $s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = 0$ and this implies that either $n = m+1$ or else $a_1 = 0$. If $n = m+1$ then G is chromatically unique by Theorem 2 of [3] (see also Theorem 3 of [4]). Hence assume that $a_1 = 0$. Note that if $E_2 = \emptyset$ or if $E_3 = \emptyset$, then $Y \cong H + \overline{K}_n$ for some bipartite graph H, which by Lemma 3.1(i), implies Y is isomorphic to G. Hence $a_2 \neq 0$ and $a_3 \neq 0$. Since $s_{p-2}(\overline{G}) - s_{p-2}(\overline{Y}) = 0$, by Lemma 2.1(ii), Y has no coincidence pair. Now, for any $Y \in \mathscr{K}^e(m+1,n,n)$, we see that $n(C_4^*,Y) = A(m,n) - en^2 + g(n)$ for some linear function $g(n)$. Since $A(m,n) = n(C_4^*,K_{m,n,n+1}) + mn(n-m)/2$, and $e = n-m$, it follows that $n(C_4^*,Y) = n(C_4^*,K_{m,n,n+1}) - en^2/2 - e^2n/2 + g(n)$.

Hence, it follows that $-en^2/2 - e^2n/2 + g(n) < 0$ if $n \geq N(m)$ for some natural number $N(m)$ (which depends on m). Consequently, $n(C_4^*,Y) < n(C_4^*,K_{m,n,n+1})$. ∎

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