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COMPOSITE CUBIC BÉZIER SURFACE WITH C^r CONTINUITY

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ABSTRACT

The Bézier surface with C^r continuity on each common boundary between two composite triangular Bézier patches is generated. The directional derivatives on the Bernstein polynomials are discussed. This is to ensure that the C^r continuity is maintained along the boundary between the piecewise polynomials. The composite patches are then generated by elevating the degree of the Bézier triangles.

Key words: Bézier triangle, directional derivatives, C^r continuity, de Casteljau's algorithm.

INTRODUCTION

The idea of Bernstein polynomials on triangular patches was first introduced by de Casteljau in 1959 (Boehm et al., 1984). De Casteljau's algorithm is commonly used for generating the graph that approximates the bivariate Bernstein polynomials. It was proved by Mühlbach in 1995 that the algorithm was indeed a particular case of the general extrapolation algorithm introduced by Brezinski in 1980 (Brezinski, 1980; Carstensen et al., 1995).

De Casteljau's algorithm will partition the Bézier triangle into three subtriangles. By repeating the process, the triangular Bézier patch will converge to the triangular Bézier surface. By considering the C^r continuity at the common boundary between two triangular Bézier patches, theorems and lemmas related to the directional derivatives on Bernstein polynomials are presented (Goldman, 1983). Based on the concept of C^r continuity, the Bézier surface with C^1 or C^2 continuity at the common boundary between two triangular Bézier patches are demonstrated.

Bézier Function Surface of Degree n on Triangular Domain

Consider the trivariate Bernstein polynomials of degree n on a triangle as

(1) $$B_{\mathbf{i}}^{n}(\mathbf{u}) = \frac{n!}{i!\,j!\,k!} u^i v^j w^k, \quad \mathbf{i} = (i, j, k), \quad i + j + k = n$$

where $\mathbf{u}$ = (u,v,w) are the barycentric coordinates with u+v+w=1 and u,v,w ≥0. We can also

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This paper is an extension of an unpublished report (Cheah, 2000).

obtain the recursive relation

(2) $$B_{\mathbf{i}}^{n}(\mathbf{u}) = uB_{\mathbf{i}-\mathbf{e}^1}^{n-1}(\mathbf{u}) + vB_{\mathbf{i}-\mathbf{e}^2}^{n-1}(\mathbf{u}) + wB_{\mathbf{i}-\mathbf{e}^3}^{n-1}(\mathbf{u})$$

where $\mathbf{e}^1$ = (1,0,0), $\mathbf{e}^2$ = (0,1,0) and $\mathbf{e}^3$ = (0,0,1) are index vectors. As the polynomials $B_{\mathbf{i}}^{n}(\mathbf{u})$ are linearly independent, they form the basis for the set Π_n of polynomials of degree less than or equal to n on a triangle. Then a Bézier function surface of degree n on the triangular domain

$$T = \{(u, v, w) \mid u, v, w \geq 0,\ u + v + w = 1\}$$

is described as

(3) $$p_n(\mathbf{u}) = \sum_{|\mathbf{i}|=n} b_{\mathbf{i}} B_{\mathbf{i}}^{n}(\mathbf{u}),\ \mathbf{u} \in T,\ \mathbf{i} = (i, j, k),\ |\mathbf{i}| = i + j + k = n$$

where $b_{\mathbf{i}}$ are the Bézier points of $p_n(\mathbf{u})$. The points $b_{\mathbf{i}}$ will form the Bézier net or the triangular Bézier patch of the triangular Bézier function surface $p_n(\mathbf{u})$ (Chen and Wang, 2002). Sometimes the Bézier net and the function surface $p_n(\mathbf{u})$ are simply termed as Bézier triangle.

Based on the relation (2), a recursive relation for generating triangular surfaces can be obtained as follows (Farin, 1981; 1982).

Theorem 1 Let $T = \{(u, v, w) \mid u, v, w \geq 0,\ u + v + w = 1\}$ and the Bézier triangle of degree n on the triangular domain T as

$$p_n(\mathbf{u}) = \sum_{|\mathbf{i}|=n} b_{\mathbf{i}} B_{\mathbf{i}}^{n}(\mathbf{u}),\quad \mathbf{i} = (i,j,k),\quad |\mathbf{i}| = i + j + k$$

We define $b_{\mathbf{i}}^{0}(\mathbf{u}) = b_{\mathbf{i}}$ and for r = 1,2,…,n,

(4) $$b_{\mathbf{i}}^{r}(\mathbf{u}) = ub_{\mathbf{i}+\mathbf{e}^1}^{r-1}(\mathbf{u}) + vb_{\mathbf{i}+\mathbf{e}^2}^{r-1}(\mathbf{u}) + wb_{\mathbf{i}+\mathbf{e}^3}^{r-1}(\mathbf{u}),\quad |\mathbf{i}| = n - r$$

Then for r = 0,1,…,n,

(5) $$p_n(\mathbf{u}) = \sum_{|\mathbf{i}|=n-r} b_{\mathbf{i}}^{r}(\mathbf{u}) B_{\mathbf{i}}^{n-r}(\mathbf{u})$$

Specifically, when r = n,

(6) $$p_n(\mathbf{u}) = b_{\mathbf{0}}^{n}(\mathbf{u})$$

Since $p_n(\mathbf{u}) = b_{\mathbf{0}}^{n}(\mathbf{u})$ when r = n, then (4) and (6) give an algorithm to compute $p_n(\mathbf{u})$. This algorithm is known as the de Casteljau's algorithm for trivariate polynomials of degree n. Figure 1 shows the geometry behind de Casteljau's algorithm. For any barycentric coordinates (u,v,w), the points $p_n(\mathbf{u})$ are determined. The points $p_n(\mathbf{u})$ will subdivide the control net of the Bézier triangle into three subtriangles.

Theorem 2 If $b^r_{\mathbf{i}}(\mathbf{u}), r = 0,1,...,n$ is defined as (4), then

(7) $$b^r_{\mathbf{i}}(\mathbf{u}) = \sum_{|\boldsymbol{\alpha}|=r} b_{\mathbf{i}+\boldsymbol{\alpha}} B^r_{\boldsymbol{\alpha}}(\mathbf{u}), \quad |\mathbf{i}| = n - r.$$

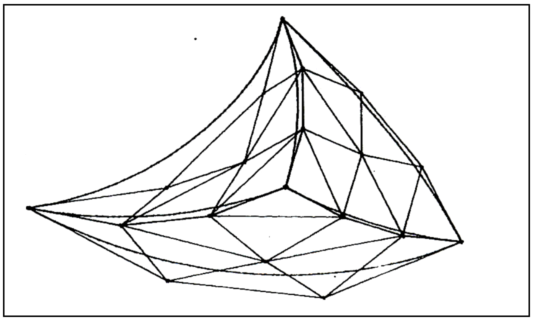

Figure 1. The formation of three subtriangles through de Casteljau's algorithm. Bézier curves are generated at the common boundaries between two adjacent patches.

DIRECTIONAL DERIVATIVES

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^+ \text{ x } \mathbb{R}^+ \text{ x } \mathbb{R}^+$ and $\mathbf{x} = (x_1, x_2, x_3)$ be the barycentric coordinates. We define (Farin, 1981)

(8) $$D^{\boldsymbol{\alpha}} f = \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}, \quad |\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \alpha_3$$

Consider **u** as the vector in the direction of two points with different barycentric coordinates. Assume that $\mathbf{u} = (u_1, u_2, u_3)$. Then $u_1+u_2+u_3= 0$. Hence, the directional derivatives of a function f in the direction of **u** can be written as

(9) $$D_{\mathbf{u}} f = u_1 \frac{\partial f}{\partial x_1} + u_2 \frac{\partial f}{\partial x_2} + u_3 \frac{\partial f}{\partial x_3}$$

As $B^1_{(1,0,0)}(\mathbf{u}) = u_1$, $B^1_{(0,1,0)}(\mathbf{u}) = u_2$ and $B^1_{(0,0,1)}(\mathbf{u}) = u_3$, we can have

(10) $$D_{\mathbf{u}} f = \sum_{|\boldsymbol{\alpha}|=1} B^1_{\boldsymbol{\alpha}}(\mathbf{u}) D^{\boldsymbol{\alpha}} f$$

Theorem 3 The directional derivative of order r for a function f in the direction of the vector $\mathbf{u} = (u_1, u_2, u_3)$, $u_1+ u_2+ u_3 = 0$, is given by

(11) $$D^r_{\mathbf{u}} f = \sum_{|\boldsymbol{\alpha}|=r} B^r_{\boldsymbol{\alpha}}(\mathbf{u}) D^{\boldsymbol{\alpha}} f, \; r = 0,1,...,n, \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^+ \text{ x } \mathbb{R}^+ \text{ x } \mathbb{R}^+$$

where $D^0 f = f$ and $D^0_{\mathbf{u}} f = f$

Proof The derivative equation (11) is true for r = 0. Assume that it is true for r. Then,

$$D_{\mathbf{u}}^{r+1} f = u_1 \frac{\partial}{\partial x_1}(D_{\mathbf{u}}^r f) + u_2 \frac{\partial}{\partial x_2}(D_{\mathbf{u}}^r f) + u_3 \frac{\partial}{\partial x_3}(D_{\mathbf{u}}^r f)$$

$$= u_1 \sum_{|\boldsymbol{\alpha}|=r} B_{\boldsymbol{\alpha}}^r(\mathbf{u}) D^{\boldsymbol{\alpha}+\mathbf{e}^1} f + u_2 \sum_{|\boldsymbol{\alpha}|=r} B_{\boldsymbol{\alpha}}^r(\mathbf{u}) D^{\boldsymbol{\alpha}+\mathbf{e}^2} f + u_3 \sum_{|\boldsymbol{\alpha}|=r} B_{\boldsymbol{\alpha}}^r(\mathbf{u}) D^{\boldsymbol{\alpha}+\mathbf{e}^3} f$$

$$= \sum_{|\boldsymbol{\alpha}|=r+1} \left[u_1 B_{\boldsymbol{\alpha}-\mathbf{e}^1}^r(\mathbf{u}) + u_2 B_{\boldsymbol{\alpha}-\mathbf{e}^2}^r(\mathbf{u}) + u_3 B_{\boldsymbol{\alpha}-\mathbf{e}^3}^r(\mathbf{u}) \right] D^{\boldsymbol{\alpha}} f$$

$$= \sum_{|\boldsymbol{\alpha}|=r+1} B_{\boldsymbol{\alpha}}^{r+1}(\mathbf{u}) D^{\boldsymbol{\alpha}} f$$

Thus (11) is true by induction. ■

Lemma 1 For $\mathbf{x} = (x_1,x_2,x_3)$, $x_1+x_2+x_3 = 1$, and $\mathbf{u} = (u_1,u_2,u_3)$, $u_1+u_2+u_3 = 0$,

(12) $$D_{\mathbf{u}}^r B_{\mathbf{i}}^n(\mathbf{x}) = \frac{n!}{(n-r)!} \sum_{|\boldsymbol{\alpha}|=r} B_{\boldsymbol{\alpha}}^r(\mathbf{u}) B_{\mathbf{i}-\boldsymbol{\alpha}}^{n-r}(\mathbf{x})$$

where $\mathbf{i} = (i_1, i_2, i_3)$ and $i_1 + i_2 + i_3 = n$.

Proof From Theorem 3,

$$D_{\mathbf{u}}^r B_{\mathbf{i}}^n(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|=r} B_{\boldsymbol{\alpha}}^r(\mathbf{u}) D^{\boldsymbol{\alpha}} B_{\mathbf{i}}^n(\mathbf{x})$$

But
$$D^{\boldsymbol{\alpha}} B_{\mathbf{i}}^n(\mathbf{x}) = \frac{\partial^{|\boldsymbol{\alpha}|}\left(\dfrac{n!}{i_1!\ i_2!\ i_3!} x_1^{i_1}\ x_2^{i_2}\ x_2^{i_2} \right)}{\partial x_1^{\alpha_1}\ \partial x_2^{\alpha_2}\ \partial x_3^{\alpha_3}}$$

$$= \frac{n!}{(i_1-\alpha_1)!\ (i_2-\alpha_2)!\ (i_3-\alpha_3)!} x_1^{i_1-\alpha_1}\ x_2^{i_2-\alpha_2}\ x_3^{i_3-\alpha_3}$$

So,

(13) $$D^{\boldsymbol{\alpha}} B_{\mathbf{i}}^n(\mathbf{x}) = \frac{n!}{(n-r)!} B_{\mathbf{i}-\boldsymbol{\alpha}}^{n-r}(\mathbf{x})$$

By substituting (13), Lemma (1) is proven. ■

Theorem 4 Assume that $p_n(\mathbf{x}) = \sum_{|\mathbf{i}|=n} b_{\mathbf{i}} B_{\mathbf{i}}^n(\mathbf{x})$. Define $b_{\mathbf{i}}^0(\mathbf{x}) = b_{\mathbf{i}}$. For r = 1,2,...,n,

(14) $$b_{\mathbf{i}}^r(\mathbf{x}) = x_1 b_{\mathbf{i}+\mathbf{e}^1}^{r-1}(\mathbf{x}) + x_2 b_{\mathbf{i}+\mathbf{e}^2}^{r-1}(\mathbf{x}) + x_3 b_{\mathbf{i}+\mathbf{e}^3}^{r-1}(\mathbf{x})$$

Then for r = 0,1,...,n, the directional derivative of order r in the direction of $\mathbf{u} = (u_1,u_2,u_3)$, $u_1+u_2+u_3 = 0$ is given by

(15) $$D_{\mathbf{u}}^{r} p_n(\mathbf{x}) = \frac{n!}{(n-r)!} \sum_{|\boldsymbol{\alpha}|=r} b_{\boldsymbol{\alpha}}^{n-r}(\mathbf{x}) B_{\boldsymbol{\alpha}}^{r}(\mathbf{u})$$

Proof Since

$$p_n(\mathbf{x}) = \sum_{|\mathbf{i}|=n} b_{\mathbf{i}} B_{\mathbf{i}}^{n}(\mathbf{x})$$

$$\Rightarrow \qquad D_{\mathbf{u}}^{r} p_n(\mathbf{x}) = \sum_{|\mathbf{i}|=n} b_{\mathbf{i}} D_{\mathbf{u}}^{r} B_{\mathbf{i}}^{n}(\mathbf{x})$$

From Lemma 1,

$$\begin{aligned} D_{\mathbf{u}}^{r} p_n(\mathbf{x}) &= \frac{n!}{(n-r)!} \sum_{|\mathbf{i}|=n} b_{\mathbf{i}} \sum_{|\boldsymbol{\alpha}|=r} B_{\boldsymbol{\alpha}}^{r}(\mathbf{u}) B_{\mathbf{i}-\boldsymbol{\alpha}}^{n-r}(\mathbf{x}) \\ &= \frac{n!}{(n-r)!} \sum_{|\mathbf{k}|=n-r} \sum_{|\boldsymbol{\alpha}|=r} b_{\mathbf{k}+\boldsymbol{\alpha}} B_{\boldsymbol{\alpha}}^{r}(\mathbf{u}) B_{\mathbf{k}}^{n-r}(\mathbf{x}), \quad \mathbf{k} = \mathbf{i} - \boldsymbol{\alpha} \\ &= \frac{n!}{(n-r)!} \sum_{|\boldsymbol{\alpha}|=r} B_{\boldsymbol{\alpha}}^{r}(\mathbf{u}) \sum_{|\mathbf{k}|=n-r} b_{\mathbf{k}+\boldsymbol{\alpha}} B_{\mathbf{k}}^{n-r}(\mathbf{x}) \\ &= \frac{n!}{(n-r)!} \sum_{|\boldsymbol{\alpha}|=r} b_{\boldsymbol{\alpha}}^{n-r}(\mathbf{x}) B_{\boldsymbol{\alpha}}^{r}(\mathbf{u}) \ \text{(from Theorem 2)} \end{aligned}$$

■

Corollary 1

(16) $$D_{\mathbf{u}}^{r} p_n(\mathbf{x}) = \frac{n!}{(n-r)!} \sum_{|\mathbf{k}|=n-r} b_{\mathbf{k}}^{r}(\mathbf{u}) B_{\mathbf{k}}^{n-r}(\mathbf{x})$$

Proof From (15),

$$D_{\mathbf{u}}^{r} p_n(\mathbf{x}) = \frac{n!}{(n-r)!} \sum_{|\mathbf{k}|=n-r} \left(\sum_{|\boldsymbol{\alpha}|=r} b_{\mathbf{k}+\boldsymbol{\alpha}} B_{\boldsymbol{\alpha}}^{r}(\mathbf{u}) \right) B_{\mathbf{k}}^{n-r}(\mathbf{x})$$

Then from Theorem 2,

$$D_{\mathbf{u}}^{r} p_n(\mathbf{x}) = \frac{n!}{(n-r)!} \sum_{|\mathbf{k}|=n-r} b_{\mathbf{k}}^{r}(\mathbf{u}) B_{\mathbf{k}}^{n-r}(\mathbf{x}) \quad ■$$

Lemma 2 Define $b_{\mathbf{i}}^{0}(\mathbf{x}) = b_{\mathbf{i}}$ for every $i_1+i_2+i_3 = n$ and for r = 1,2,...,n ,

$$b_{\mathbf{i}}^{r}(\mathbf{x}) = x_1 b_{\mathbf{i}+\mathbf{e}^1}^{r-1}(\mathbf{x}) + x_2 b_{\mathbf{i}+\mathbf{e}^2}^{r-1}(\mathbf{x}) + x_3 b_{\mathbf{i}+\mathbf{e}^3}^{r-1}(\mathbf{x})$$

Then for any direction $\mathbf{u}$, $u_1+u_2+u_3 = 0$,

(17) $$D_{\mathbf{u}}^{r} b_{\mathbf{i}}^{r}(\mathbf{x}) = r!\, b_{\mathbf{i}}^{r}(\mathbf{u}), \quad r = 0,1,\ldots,n.$$

Proof From Theorem 2, for r = 0,1,...,n ,

$$b^r_{\mathbf{i}}(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|=r} b_{\mathbf{i}+\boldsymbol{\alpha}} B^r_{\boldsymbol{\alpha}}(\mathbf{x})$$

So,

$$D^r_{\mathbf{u}} b^r_{\mathbf{i}}(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|=r} b_{\mathbf{i}+\boldsymbol{\alpha}} D^r_{\mathbf{u}} B^r_{\boldsymbol{\alpha}}(\mathbf{x})$$

From Lemma 1,

$$D^r_{\mathbf{u}} B^r_{\boldsymbol{\alpha}}(\mathbf{x}) = r!\sum_{|\mathbf{k}|=r} B^r_{\mathbf{k}}(\mathbf{u}) B^0_{\boldsymbol{\alpha}-\mathbf{k}}(\mathbf{x})$$

$$= r!\, B^r_{\boldsymbol{\alpha}}(\mathbf{u})$$

since

$$B^0_{\boldsymbol{\alpha}-\mathbf{k}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \boldsymbol{\alpha} = \mathbf{k} \\ 0 & \text{if } \boldsymbol{\alpha} \neq \mathbf{k} \end{cases}$$

Then,

$$D^r_{\mathbf{u}} b^r_{\mathbf{i}}(\mathbf{x}) = r!\sum_{|\boldsymbol{\alpha}|=r} b_{\mathbf{i}+\boldsymbol{\alpha}} B^r_{\boldsymbol{\alpha}}(\mathbf{u})$$

$$= r!\, b^r_{\mathbf{i}}(\mathbf{u}) \qquad \text{(from Theorem 2)}$$

■

Let us consider a triangle $[T_1, T_2, T_3]$ where $b_{(n,0,0)}$, $b_{(0,n,0)}$ and $b_{(0,0,n)}$ are the Bézier points at T_1, T_2, T_3 respectively. Let $\mathbf{u_1}$, $\mathbf{u_2}$ and $\mathbf{u_3}$ be the vectors parallel to the directions of $\overrightarrow{T_2T_3}$, $\overrightarrow{T_3T_1}$ and $\overrightarrow{T_1T_2}$ respectively. We define

(i) $\Delta_{2,3} b_{(\alpha, i_2, i_3)} = b_{(\alpha, i_2, i_3+1)} - b_{(\alpha, i_2+1, i_3)}$ and

(18) $\Delta^r_{2,3} = \Delta_{2,3}\Delta^{r-1}_{2,3}$, α constant,

(ii) $\Delta_{3,1} b_{(i_1, \beta, i_3)} = b_{(i_1+1, \beta, i_3)} - b_{(i_1, \beta, i_3+1)}$ and

(19) $\Delta^r_{3,1} = \Delta_{3,1}\Delta^{r-1}_{3,1}$, β constant,

(iii) $\Delta_{1,2} b_{(i_1, i_2, \gamma)} = b_{(i_1, i_2+1, \gamma)} - b_{(i_1+1, i_2, \gamma)}$ and

(20) $\Delta^r_{1,2} = \Delta_{1,2}\Delta^{r-1}_{1,2}$, γ constant,

whereby for $\mathbf{i} = (i_1, i_2, i_3)$, $i_1+i_2+i_3 = n$, $b_{\mathbf{i}}$ are Bézier points that will form the triangle $[T_1, T_2, T_3]$.

Lemma 3 On the triangle $[T_1, T_2, T_3]$, for $r = 0, 1, \ldots, n$ and α, β, γ are constants,

(i) if $\mathbf{u_1} = \overrightarrow{T_2T_3}$, then

$$D^r_{\mathbf{u_1}} b^r_{(\alpha, i_2, i_3)}(\mathbf{x}) = r!\, b^r_{(\alpha, i_2, i_3)}(\mathbf{u_1}) = r!\,\Delta^r_{2,3} b_{(\alpha, i_2, i_3)}$$

(ii) if $\mathbf{u_2} = \overrightarrow{T_3T_1}$, then

$$D^r_{\mathbf{u_2}} b^r_{(i_1,\beta,i_3)}(\mathbf{x}) = r!\Delta^r_{3,1} b_{(i_1,\beta,i_3)}$$

(iii) if $\mathbf{u_3} = \overrightarrow{T_1T_2}$, then

$$D^r_{\mathbf{u_3}} b^r_{(i_1,i_2,\gamma)}(\mathbf{x}) = r!\Delta^r_{1,2} b_{(i_1,i_2,\gamma)}$$

Proof It is sufficient to prove one of the cases due to the symmetric property. Let us consider case (i).
The equation in case (i) is true for r = 0 since

$$D^0_{\mathbf{u_1}} b^0_{(\alpha,i_2,i_3)}(\mathbf{x}) = b^0_{(\alpha,i_2,i_3)}(\mathbf{x}) = b_{(\alpha,i_2,i_3)}$$
$$= \Delta^0_{2,3} b_{(\alpha,i_2,i_3)}$$

If the equation is true for r, then from Lemma 2,

$$D^{r+1}_{\mathbf{u_1}} b^{r+1}_{(\alpha,i_2,i_3)}(\mathbf{x}) = (r+1)!\, b^{r+1}_{(\alpha,i_2,i_3)}(\mathbf{u_1})$$

Here $\mathbf{u_1}$ = (0 , –1 , 1) . That means

$$(r+1)!\, b^{r+1}_{(\alpha,i_2,i_3)}(\mathbf{u_1})$$
$$= (r+1)!\left(0 - b^r_{(\alpha,i_2+1,i_3)}(\mathbf{u_1}) + b^r_{(\alpha,i_2,i_3+1)}(\mathbf{u_1})\right)$$
$$= (r+1)!\left(\Delta^r_{2,3} b_{(\alpha,i_2,i_3+1)} - \Delta^r_{2,3} b_{(\alpha,i_2+1,i_3)}\right)$$
$$= (r+1)!\,\Delta^{r+1}_{2,3} b_{(\alpha,i_2,i_3)}$$

Thus Lemma 3 is true by mathematical induction. ■

Composite Cubic Bézier Patches With C^r Continuity

By considering any two arbitrary triangles $[T_1, T_2, T_3]$ and $[\hat{T}_1, T_2, T_3]$, let $\mathbf{x} = (x_1, x_2, x_3)$ and $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ represent a point on $[T_1, T_2, T_3]$ and $[\hat{T}_1, T_2, T_3]$ respectively.

The points at the common boundary T_2T_3 have coordinates $(0, x_2, x_3)$ and $(0, \hat{x}_2, \hat{x}_3)$ relative to $[T_1, T_2, T_3]$ and $[\hat{T}_1, T_2, T_3]$ respectively. Assume that $\hat{T}_1$ has barycentric coordinates (t_1, t_2, t_3) on $[T_1, T_2, T_3]$. This implies that

$$\hat{T}_1 = t_1 T_1 + t_2 T_2 + t_3 T_3$$

Given $\hat{\mathbf{x}}$ as the barycentric coordinates in $[\hat{T}_1, T_2, T_3]$, then

$$\mathbf{x} = \hat{\mathbf{x}}$$

$$\Leftrightarrow \quad \begin{cases} x_1 &= \hat{x}_1 t_1 \\ x_2 &= \hat{x}_1 t_2 + \hat{x}_2 \\ x_3 &= \hat{x}_1 t_3 + \hat{x}_3 \end{cases}$$

This means that any point P with barycentric coordinates $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ on $[\hat{T}_1, T_2, T_3]$ will have its equivalent $\mathbf{x} = (\hat{x}_1 t_1, \hat{x}_1 t_2 + \hat{x}_2, \hat{x}_1 t_3 + \hat{x}_3)$ on $[T_1, T_2, T_3]$, where $\hat{T}_1 = (t_1, t_2, t_3)$ towards $[T_1, T_2, T_3]$. Specifically, if P is situated at the common boundary T_2T_3, then its barycentric coordinates are $\mathbf{x} = (0, x_2, x_3) = \hat{\mathbf{x}}$.

Let $b_{\mathbf{i}}$ and $\hat{b}_{\mathbf{i}}$ be the Bézier points towards $[T_1, T_2, T_3]$ and $[\hat{T}_1, T_2, T_3]$ respectively. Then, for polynomials

$$p_n(\mathbf{x}) = \sum_{|\mathbf{i}|=n} b_{\mathbf{i}} B_{\mathbf{i}}^n(\mathbf{x}) \text{ towards } [T_1, T_2, T_3]$$

and

$$\hat{p}_n(\hat{\mathbf{x}}) = \sum_{|\mathbf{i}|=n} \hat{b}_{\mathbf{i}} B_{\mathbf{i}}^n(\hat{\mathbf{x}}) \text{ towards } [\hat{T}_1, T_2, T_3]$$

we are interested in finding the relationship between $b_{\mathbf{i}}$ and $\hat{b}_{\mathbf{i}}$ so that $p_n(\mathbf{x}) = \hat{p}_n(\hat{\mathbf{x}})$ for all $\mathbf{x}$ and $\hat{\mathbf{x}}$ that represents the same barycentric coordinates. For example,

$$p_1(\mathbf{x}) = \hat{p}_1(\hat{\mathbf{x}})$$

$$\Leftrightarrow \quad \begin{cases} \hat{b}_{(1,0,0)} = b^1_{(0,0,0)}(\mathbf{t}) \\ \hat{b}_{(0,1,0)} = b_{(0,1,0)} \\ \hat{b}_{(0,0,1)} = b_{(0,0,1)} \end{cases}$$

and

$$p_2(\mathbf{x}) = p_2(\hat{\mathbf{x}})$$

$$\Leftrightarrow \quad \begin{cases} \hat{b}_{(2,0,0)} = b^2_{(0,0,0)}(\mathbf{t}) \\ \hat{b}_{(0,2,0)} = b_{(0,2,0)} \\ \hat{b}_{(0,0,2)} = b_{(0,0,2)} \\ \hat{b}_{(1,1,0)} = b^1_{(0,1,0)}(\mathbf{t}) \\ \hat{b}_{(1,0,1)} = b^1_{(0,0,1)}(\mathbf{t}) \\ \hat{b}_{(0,1,1)} = b_{(0,1,1)} \end{cases}$$

Theorem 5 Let $p_n(\mathbf{x}) = \sum_{|\mathbf{i}|=n} b_{\mathbf{i}} B_{\mathbf{i}}^n(\mathbf{x})$ be a polynomial that is defined under barycentric coordinates $\mathbf{x}$ towards $[T_1, T_2, T_3]$ and $\hat{p}_n(\hat{\mathbf{x}}) = \sum_{|\mathbf{i}|=n} \hat{b}_{\mathbf{i}} B_{\mathbf{i}}^n(\hat{\mathbf{x}})$ be another polynomial that is defined under barycentric coordinates $\hat{\mathbf{x}}$ towards $[\hat{T}_1, T_2, T_3]$. Then the polynomials $p_n(\mathbf{x})$ and $\hat{p}_n(\hat{\mathbf{x}})$ have C^r continuity along the common boundary T_2T_3

(21)

$$\Leftrightarrow b^{\rho}_{(0,i_2,i_3)}(t_1, t_2 - 1, t_3) = \hat{b}^{\rho}_{(0,i_2,i_3)}(1, -1, 0) \quad \forall \rho = 0,1,...,r, \; i_2 + i_3 = n - \rho, \; 0 \le r \le n$$

where (t_1, t_2, t_3) are the barycentric coordinates of $\hat{T}_1$ on the triangle $[T_1, T_2, T_3]$.

Proof The polynomials $p_n(\mathbf{x})$ and $\hat{p}_n(\hat{\mathbf{x}})$ have C^r continuity at T_2T_3 if and only if for any two linearly independent vectors $\mathbf{u}_v = \hat{\mathbf{u}}_v, \mathbf{v} = (1,2)$,

(22)

$$D^{\rho}_{\mathbf{u}} p_n(0, x_2, x_3) = D^{\rho}_{\hat{\mathbf{u}}} \hat{p}_n(0, x_2, x_3) \;\; \forall (0, x_2, x_3) \in T_2T_3, \quad \rho = 0,1,...,r, \; 0 \le r \le n$$

$$\Leftrightarrow \quad \sum_{|\mathbf{i}|=n-\rho} b^{\rho}_{\mathbf{i}}(\mathbf{u}_v) B^{n-\rho}_{\mathbf{i}}(\mathbf{x}) = \sum_{|\mathbf{i}|=n-\rho} \hat{b}^{\rho}_{\mathbf{i}}(\hat{\mathbf{u}}_v) B^{n-\rho}_{\mathbf{i}}(\hat{\mathbf{x}}), \quad \hat{\mathbf{x}} = \mathbf{x} = (0, x_2, x_3)$$

$$\Leftrightarrow \quad b^{\rho}_{(0,i_2,i_3)}(\mathbf{u}_v) = \hat{b}^{\rho}_{(0,i_2,i_3)}(\hat{\mathbf{u}}_v), \; \rho = 0,1,...,r, \; 0 \le r \le n$$

Specifically,

$$\mathbf{u_1} = \overrightarrow{T_2T_3} = (0, -1, 1) = \hat{\mathbf{u}}_\mathbf{1},$$

$$\mathbf{u_2} = \overrightarrow{T_2\hat{T}_1} = (t_1, t_2 - 1, t_3),$$

$$\hat{\mathbf{u}}_\mathbf{2} = (1, -1, 0) = \overrightarrow{\hat{T}_2\hat{T}_1}$$

Therefore, we obtain

(23) $\quad b^{\rho}_{(0,i_2,i_3)}(0, -1, 1) = \hat{b}^{\rho}_{(0,i_2,i_3)}(0, -1, 1)$

(24)

$$b^{\rho}_{(0,i_2,i_3)}(t_1, t_2 - 1, t_3) = \hat{b}^{\rho}_{(0,i_2,i_3)}(1, -1, 0) \;\; \forall \rho = 0,1,...,r, \; i_2 + i_3 = n - \rho, \; 0 \le r \le n$$

From Lemma 3, the equation (23) yields

$$\Delta^{\rho}_{2,3} b_{(0,i_2,i_3)} = \Delta^{\rho}_{2,3} \hat{b}_{(0,i_2,i_3)}, \; i_2 + i_3 = n - \rho$$

$$\Leftrightarrow \quad b_{(0,i_2,i_3)} = \hat{b}_{(0,i_2,i_3)}, \; i_2 + i_3 = n$$

and this is the condition for (24) when $\rho = 0$. Thus, the polynomials $p_n(\mathbf{x})$ and $\hat{p}_n(\hat{\mathbf{x}})$ have C^r continuity on T_2T_3 if and only if

$$b_{(0,i_2,i_3)}(t_1, t_2 - 1, t_3) = \hat{b}_{(0,i_2,i_3)}(1, -1, 0)$$

for $\rho = 0,1,...,r$, $0 \le r \le n$ and $i_2 + i_3 = n - \rho$. ■

Corollary 2 For every $\rho = 0,1,\ldots,r$, $0 \le r \le n$,

$$p_n(\mathbf{x}) = \hat{p}_n(\hat{\mathbf{x}})$$

(25) $$\Leftrightarrow \hat{b}_{(\rho,i_2,i_3)} = b^{\rho}_{(0,i_2,i_3)}(\mathbf{t}),\ i_2 + i_3 = n - \rho$$

where $\mathbf{t} = (t_1, t_2, t_3)$, $t_1 + t_2 + t_3 = 1$ are barycentric coordinates for $\hat{T}_1$ on $[T_1, T_2, T_3]$.

Proof From Lemma 3, the equation (21) is equivalent with

$$D^{\rho}_{\mathbf{t}'} b^{\rho}_{(0,i_2,i_3)}(\mathbf{x}) = D^{\rho}_{\mathbf{t}''} \hat{b}^{\rho}_{(0,i_2,i_3)}(\hat{\mathbf{x}}),\ \rho = 0,\ldots,r,\ 0 \le r \le n$$

where $\mathbf{t}' = (t_1, t_2 - 1, t_3)$ and $\mathbf{t}'' = (1,-1,0)$.

Since $\mathbf{t}'$ and $\mathbf{t}''$ have the same directions, therefore

(26) $$b^{\rho}_{(0,i_2,i_3)}(\mathbf{x}) = \hat{b}^{\rho}_{(0,i_2,i_3)}(\hat{\mathbf{x}})$$

But equation (26) is also true for $\mathbf{x} = \mathbf{t} = (t_1, t_2, t_3)$ and $\hat{\mathbf{x}} = (1,0,0)$. So,

$$b^{\rho}_{(0,i_2,i_3)}(\mathbf{t}) = \hat{b}^{\rho}_{(0,i_2,i_3)}(1,0,0)$$

$$= \hat{b}_{(\rho,i_2,i_3)},$$

for $\rho = 0,1,\ldots,r$, $0 \le r \le n$, $i_2 + i_3 = n - \rho$. ■

Implications of Corollary 2 :

(a) When $\hat{\mathbf{T}}_1 = \mathbf{t}$ is inside the triangle $[T_1, T_2, T_3]$, then $\hat{T}_1$ will divide the triangle $[T_1, T_2, T_3]$ into three subtriangles. The points $b^{n}_{(0,i_2,i_3)}(\mathbf{t})$ that are obtained will be the same as the required coefficients. See Figure 2.

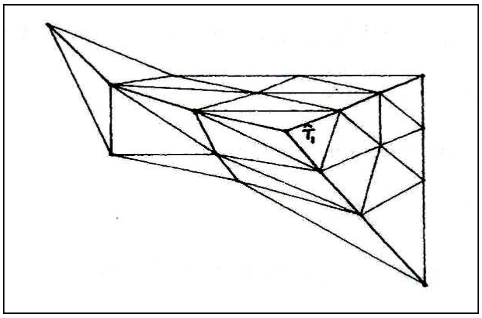

Figure 2. Partitioning of $[T_1, T_2, T_3]$ by $\hat{\mathbf{T}}_1$ into three subtriangles when $\hat{\mathbf{T}}_1$ is inside the triangle $[T_1, T_2, T_3]$

(b) When $\hat{\mathbf{T}}_1$ is outside the triangle $[T_1, T_2, T_3]$, then for the given Bézier points $b_{\mathbf{i}}$, the points $\hat{b}_{\mathbf{i}}$ have to satisfy the C^r continuity at the common boundary T_2T_3 between the two surfaces $p_n(\mathbf{x})$ and $\hat{p}_n(\hat{\mathbf{x}})$. See Figure 3.

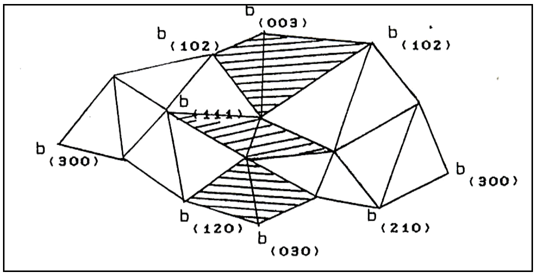

Figure 3. Two adjacent triangles form a smooth Quadrilateral with C^1 continuity

Example Let n = 3, r = l. Assume that $\mathbf{t} = (t_1, t_2, t_3)$ is situated outside the control triangular Bézier patch.

For $\rho = 0$, equation (25) implies that

$$\hat{b}^{\rho}_{(0,i_2,i_3)} = b_{(0,i_2,i_3)}, \quad i_2 + i_3 = 3$$

For $\rho = 1$,

$$\begin{aligned}\hat{b}^{\rho}_{(1,i_2,i_3)} &= b^{1}_{(0,i_2,i_3)}(\mathbf{t}), \qquad i_2 + i_3 = 2\\ &= t_1 b_{(1,i_2,i_3)} + t_2 b_{(0,i_2+1,i_3)} + t_3 b_{(0,i_2,i_3+1)}\end{aligned}$$

Therefore for C^1 continuity, the quadrilaterals formed by the two adjacent triangles at the common boundary between the two triangular Bézier patches are smooth planes.

Figure 4(left) shows the combined composite patches with control net at the center accompanied by two composite patches with C^1 and C^2 continuities on the right and left of the control net respectively. Figure 4(right) is the improvement of Figure 4(left) by raising 8 degrees. A closed surface can be obtained by setting the barycentric coordinates based on some end conditions as seen in Figures 5 and 6.

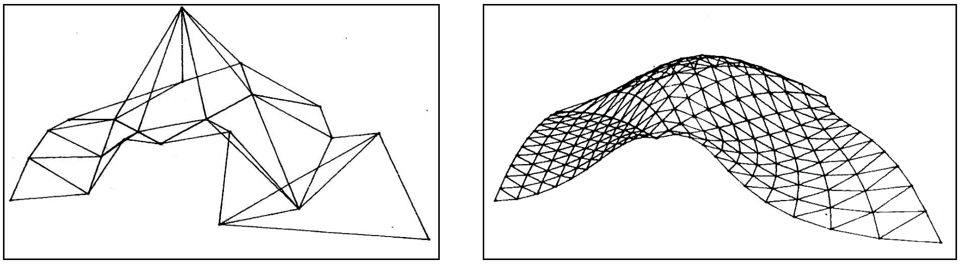

Figure 4. (Left) The combined composite cubic patches with control net at the center accompanied by two composite patches with C^1 and C^2 continuities on the right and left of the control net respectively. (Right) The effect of raising by 8 degrees (Cohen et al., 1985).

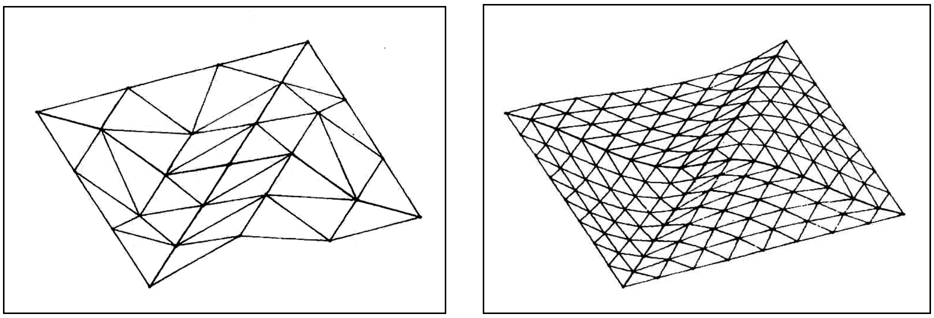

Figure 5. (Left) The control composite cubic Bézier patches with C^1 continuity at each common boundary. (Right) The effect of 5 degrees elevation.

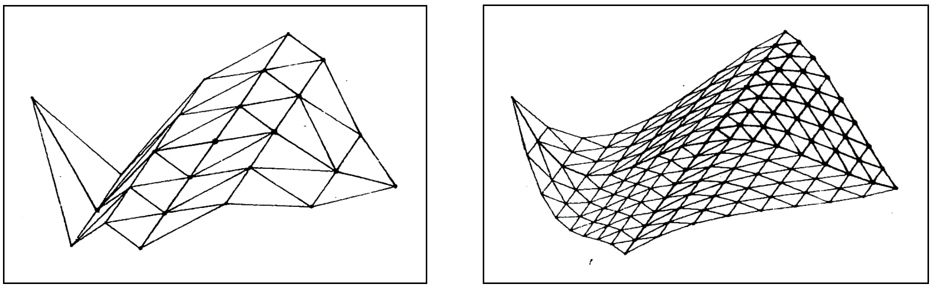

Figure 6. (Left) The control composite cubic Bézier patches with C^2 continuity at each common boundary. (Right) The effect of 5 degrees elevation.

CONCLUSION

The concept of triangular spline has improved the flexibility in designing objects. The wide application of triangular spline in solid modeling and texture mapping has always been a great challenge for computer graphics and virtual environments (Sheffer et al., 2002; Chandrajit et al., 2003). Hence, the study in the degree of continuity is important for designing objects that are formed by piecewise patches. As we can see, the C^r continuity at the common boundaries between triangular patches has made the composition of surfaces possible (Stephen, 2002). Future development will see the possibility of controlling the curve and surfaces based on the interval parameterization concept whereby developable objects can be generated without fail (Chen and Deng, 2003).

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