Confidence Intervals for Multivariate Value at Risk

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Abstract: Confidence intervals for the $\gamma$-quantile of a linear combination of $N$ non-normal variates with a linear dependence structure would be useful to the financial institutions as the intervals enable the accuracy of the value at risk (VaR) of a portfolio of investments to be quantified. Presently, we construct 100(1-$\alpha$) % confidence intervals for the $\gamma$-quantile using the procedures based on bootstrap, normal approximation and hypothesis testing. We show that the method based on hypothesis testing produces confidence interval which is more satisfactory than those found by using bootstrap or normal approximation.

Keywords: non-normal variates, multivariate value at risk, $\gamma$-quantile, confidence interval.

1. Introduction

Consider a portfolio consisting of $N$ stocks. The absolute value of the $\gamma$-quantile of the return of the portfolio is called the value at risk (VaR) of the portfolio. VaR has been frequently used by commercial and investment banks to capture the potential loss in value of their traded portfolios from adverse market movements over a specified period.

To evaluate VaR in the multivariate situation where $N$ stocks are involved, we usually begin with the evaluation of a multivariate distribution for the $N$ stocks. A common approach is to fit the data on returns by the multivariate normal model. However, it should be noted that the normal distribution does not fit the multivariate distribution of return very often, where the distribution of stock return very often also has fat tails and narrow waist. As the distribution of stock return very often also has fat tails and narrow waist, and the returns of different stocks are usually correlated, the distribution of the portfolio return given by Equation (1) can be used to model the joint distribution of the returns of $N$ stocks. For a portfolio of $N$ stocks, the portfolio return can be represented by $R$ given by Equation (2).

After finding an estimate for the VaR, it is usually desirable to access the accuracy of the VaR estimate by constructing a confidence interval for the VaR.

The layout of the paper is as follows. In Sections 2, 3 and 4, we describe respectively the procedures based on bootstrap, normal approximation and hypothesis testing for finding a confidence interval for the VaR. In Section 5, we compare the performance of the three methods for constructing confidence intervals for the VaR. In Section 6, we give an example which shows that multivariate quadratic-normal distribution is able to fit a real data set obtained from the Kuala Lumpur Stock Exchange.

2. Bootstrap Confidence Interval for $\gamma$-Quantile

First, let $(\tilde{r}_{1j}, \tilde{r}_{2j}, \cdots, \tilde{r}_{nj})$ be the $j$th observed value of $\mathbf{R}$, $j = 1, 2, \cdots, n$. From the $n$ observed values $(\tilde{r}_{1j}, \tilde{r}_{2j}, \cdots, \tilde{r}_{nj})$, $j = 1, 2, \cdots, n$, we first compute the $(k, l)$ entry of the matrix $\mathbf{V}$ of the estimated variance-covariance of $\mathbf{R}$ as shown below:

\[ v_{kl} = \frac{1}{n} \sum_{j=1}^{n} r_{kj} r_{lj} - \tilde{\mu}_k \tilde{\mu}_l \quad \text{where} \quad \tilde{\mu}_k = \frac{1}{n} \sum_{j=1}^{n} r_{kj}. \]

We next compute $\tilde{\mathbf{A}} = [\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n]$ where $\tilde{a}_i$ is the $i$th eigenvector of $\mathbf{V}$, and $|| \tilde{a}_i || = 1$. By using $\tilde{\mathbf{A}}$, we compute

\[
\begin{bmatrix}
S_{1j} \\
S_{2j} \\
\vdots \\
S_{nj}
\end{bmatrix} = \tilde{\mathbf{A}}^T \begin{bmatrix}
\tilde{r}_{1j} - \tilde{\mu}_1 \\
\tilde{r}_{2j} - \tilde{\mu}_2 \\
\vdots \\
\tilde{r}_{nj} - \tilde{\mu}_n
\end{bmatrix},
\]

where $\tilde{\mathbf{A}}$ is an $N \times 1$ vector of constants and $\mathbf{R} = (R_1, R_2, \cdots, R_N)^T$ is an $N \times 1$ vector given by

\[
\mathbf{R} = \tilde{\mathbf{A}} \mathbf{w} + \tilde{\mathbf{A}} \mathbf{w}^2,
\]

\[ w = (w_1, w_2, \cdots, w_N)^T \] a vector of constants with $\sum_{i=1}^{N} w_i = 1$ and $R = \sum_{i=1}^{N} w_i R_i$.

When $\lambda_{12} = -1$ and $\lambda_{12}$ is large, the distribution of the random variable $S_i$ will have fat tails and narrow waist. As the matrix $\mathbf{A}$ represents an orthogonal transformation, and the vector $\mathbf{w}$, on the other hand represents a translation, the distribution of $R_i$ will also have fat tails and narrow waist. As the distribution of stock return very often also has fat tails and narrow waist, and the returns of different stocks are usually correlated, the distribution of the portfolio return can be represented by $R$ given by Equation (2).

Let $F_R$ be the cumulative distribution function of $R$ and assume that the $\gamma$-quantile, $Q_R = F_R^{-1}(\gamma)$, is uniquely defined. When $\gamma$ is small, the absolute value of $Q_R$ will represent the VaR which has a confidence level of 100(1-$\gamma$)%. After finding an estimate for the VaR, it is usually desirable to access the accuracy of the VaR estimate by constructing a confidence interval for the VaR.
By using the constrained maximum likelihood procedure [16], we find the quadratic-normal distributions \( QN(\hat{\mu}, \hat{\lambda}) \) and \( QN(\mu, \lambda) \) which fit \( s_{1b}, s_{2b}, \ldots, s_{nb} \) and the \( n \) observed values of \( R \). Let \( z_p \) be the \((1 - \gamma)\)-quantile of the standard normal distribution. An estimate of the \( \gamma \)-quantile of \( R \) is then given by

\[
\hat{Q} = \hat{\mu} + \hat{\lambda}_1 (z_{\gamma}) + \hat{\lambda}_2 [(\hat{\lambda}_3 (z_{\gamma})^2 - \frac{1 + \hat{\lambda}_3}{2}]
\]

Next, we generate \( B \) values of \((\hat{\nu}_{1b}, \hat{\nu}_{2b}, \ldots, \hat{\nu}_{Nb}) \), \( j = 1, 2, \ldots, n \), using

\[
\begin{bmatrix}
\hat{\nu}_{1b} \\
\hat{\nu}_{2b} \\
\vdots \\
\hat{\nu}_{Nb}
\end{bmatrix} = \begin{bmatrix}
\hat{\mu}_1 \\
\hat{\mu}_2 \\
\vdots \\
\hat{\mu}_N
\end{bmatrix} + \begin{bmatrix}
\hat{\lambda}_1 \\
\hat{\lambda}_2 \\
\vdots \\
\hat{\lambda}_N
\end{bmatrix} \begin{bmatrix}
\hat{\lambda}_3 \\
\hat{\lambda}_4 \\
\vdots \\
\hat{\lambda}_N
\end{bmatrix}
\]

where \( \hat{\lambda}_j \sim QN(\hat{\mu}, \hat{\lambda}) \). By using the constrained maximum likelihood procedure, we find the quadratic-normal distribution \( QN(\hat{\mu}, \hat{\lambda}) \) which fits the values \( \hat{\nu}_j = \sum_{i=1}^{n} \hat{w}_{ij} \), \( j = 1, 2, \ldots, n \). Next let

\[
\hat{Q} = \hat{\mu} + \hat{\lambda}_1 (z_{\gamma}) + \hat{\lambda}_2 [(\hat{\lambda}_3 (z_{\gamma})^2 - \frac{1 + \hat{\lambda}_3}{2}]
\]

be the estimated quantile, and \( QN(\hat{\mu}, \hat{\lambda}) \) the quadratic-normal distribution which fits the \( B \) values of \( \hat{Q} \).

The approximately-100\((1 - \alpha)\)% bootstrap confidence interval for the \( \gamma \)-quantile is then given by \( \left[ Q_L, Q_U \right] \) where

\[
Q_L = \hat{\mu} + \hat{\lambda}_1 (z_{\alpha/2}) + \hat{\lambda}_2 [(\hat{\lambda}_3 (z_{\alpha/2})^2 - \frac{1 + \hat{\lambda}_3}{2}]
\]

and

\[
Q_U = \hat{\mu} + \hat{\lambda}_1 (z_{1 - \alpha/2}) + \hat{\lambda}_2 [(z_{\alpha/2})^2 - \frac{1 + \hat{\lambda}_3}{2}].
\]

3. Confidence Intervals based on Normal Approximation

From the \( B \) values of \( \hat{Q}^{(1)}, \hat{Q}^{(2)}, \ldots, \hat{Q}^{(B)} \) of \( \hat{Q} \) in Section 2, we can find the estimated variance \( \hat{s}^2 = \frac{1}{B - 1} \sum_{b=1}^{B} (\hat{Q}^{(b)} - \hat{Q})^2 \) where \( \hat{Q} = \frac{1}{B} \sum_{b=1}^{B} \hat{Q}^{(b)} \). Then the approximately-100\((1 - \alpha)\)% confidence interval based on normal approximation for the \( \gamma \)-quantile is

\[
\left[ \hat{Q} - z_{\alpha/2} \hat{s} \hat{Q}, \hat{Q} + z_{\alpha/2} \hat{s} \hat{Q} \right].
\]

4. Procedure based on Hypothesis Testing

Consider the problem of testing \( H_0: \gamma = \gamma^{(0)} \) against \( H_1: \gamma \neq \gamma^{(0)} \). Suppose we test the above \( H_0 \) by using the decision rule “Accept \( H_0 \) at the \( \alpha \) level if \( Q^{(0)} \leq Q \leq Q^{(0)} \), where \( Q^{(0)} \) and \( Q^{(0)} \) are respectively the 100\((\alpha/2)\)% and 100\((1 - \alpha/2)\)% points of the quadratic-normal distribution which is used to fit the \( B \) values of \( \hat{Q} \) obtained when the \( B \) values of \( \left(\hat{\nu}_{1b}, \hat{\nu}_{2b}, \ldots, \hat{\nu}_{Nb}\right) \) are generated using

\[
\begin{bmatrix}
\hat{\nu}_{1b} \\
\hat{\nu}_{2b} \\
\vdots \\
\hat{\nu}_{Nb}
\end{bmatrix} = \begin{bmatrix}
\hat{\mu}_1 \\
\hat{\mu}_2 \\
\vdots \\
\hat{\mu}_N
\end{bmatrix} + \begin{bmatrix}
\hat{\lambda}_1 \\
\hat{\lambda}_2 \\
\vdots \\
\hat{\lambda}_N
\end{bmatrix} \begin{bmatrix}
\hat{\lambda}_3 \\
\hat{\lambda}_4 \\
\vdots \\
\hat{\lambda}_N
\end{bmatrix}
\]

where \( \hat{\lambda}_j \sim QN(\hat{\mu}^{(m)}, \hat{\lambda}^{(m)}) \). An approximately-100\((1 - \alpha)\)% confidence interval for the \( \gamma \)-quantile of \( R \) is now given by \( \left[ Q^{(0)}, Q^{(0)} \right] \). The null hypothesis that \( \gamma = \gamma^{(0)} \) is accepted at the \( \alpha \) level.

5. Numerical Examples

Figure 1 shows 100 simulated bootstrap confidence intervals for the \( \gamma \)-quantile of \( R \) when \( n = 50 \) and the value of \( \left(\mu_1, \lambda_1, \mu_2, \lambda_2, A\right) \) is given. In the figure, the upper limits of the 100 confidence intervals have been arranged in an ascending order.

Figures 2 and 3 show 100 possible confidence intervals based on normal approximation and hypothesis testing. As in Figure 1, the upper limits of the 100 confidence intervals have been arranged in an ascending order.

Figures 1 – 3 show that the estimated coverage probability of the confidence interval based on hypothesis testing is closer to the target value 0.95 than those of the bootstrap confidence interval and the confidence interval based on normal approximation.

Figure 1. 100 simulated bootstrap confidence intervals for \( \gamma \)-quantile when \( \mu_1 = 0, \lambda_1 = (0.32, 0.68, 0.065), \mu_2 = 0, \lambda_2 = (0.378, 0.639, 0.073) \) and \( A = (0.3090, 0.9511) \) estimated coverage probability = 0.82. Average length = 2.3945 \( \hat{Q} \) -- estimate of \( \gamma \)-quantile, \( Q \) -- true value of \( \gamma \)-quantile.
The length of the confidence interval should be approximately 0.0218. This can be found in Table 1 which displays the estimated coverage probabilities and average lengths for 10 values of quantile, $\hat{Q}$—estimate of $\gamma$-quantile, $Q_\gamma$—true value of $\gamma$-quantile.

Further comparison of the 3 types of confidence intervals can be found in Table 1 which displays the estimated coverage probabilities and average lengths for 10 values of ($\mu_1, \lambda_1, \mu_2, \lambda_2, A$). The 10 values of $\lambda_1$ and $\lambda_2$ are displayed in Table 2. The measures of skewness and kurtosis ($m_3$ and $m_4$) of the quadratic-normal distribution with the given $\lambda_i$ are also included in Table 2. Table 1 shows that the coverage probability of the confidence interval based on hypothesis testing is closer to the target value 0.95 than those of the bootstrap confidence interval and confidence interval based on normal approximation.

Table 1 also shows that the average length of the confidence interval based on the hypothesis testing is longer than those of the bootstrap confidence interval and confidence interval based on normal approximation. This is not surprising because in order to have a larger coverage probability, the length of the confidence interval should be made longer.

Table 1. Estimated coverage probabilities and average lengths of confidence intervals for $\gamma$-quantile

<table>
<thead>
<tr>
<th>No</th>
<th>BTP</th>
<th>NAP</th>
<th>HTP</th>
<th>BTL</th>
<th>NAL</th>
<th>HTL</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.82</td>
<td>0.89</td>
<td>0.91</td>
<td>2.39207</td>
<td>2.42961</td>
<td>2.918</td>
</tr>
<tr>
<td>2</td>
<td>0.8</td>
<td>0.8</td>
<td>0.88</td>
<td>2.45177</td>
<td>2.49157</td>
<td>3.019</td>
</tr>
<tr>
<td>3</td>
<td>0.79</td>
<td>0.82</td>
<td>0.93</td>
<td>2.31436</td>
<td>2.35217</td>
<td>2.9195</td>
</tr>
<tr>
<td>4</td>
<td>0.72</td>
<td>0.72</td>
<td>0.83</td>
<td>2.17881</td>
<td>2.21492</td>
<td>2.634</td>
</tr>
<tr>
<td>5</td>
<td>0.91</td>
<td>0.89</td>
<td>0.94</td>
<td>0.76893</td>
<td>0.76353</td>
<td>1.214</td>
</tr>
<tr>
<td>6</td>
<td>0.85</td>
<td>0.86</td>
<td>0.91</td>
<td>1.26915</td>
<td>1.29416</td>
<td>1.8135</td>
</tr>
<tr>
<td>7</td>
<td>0.79</td>
<td>0.85</td>
<td>0.87</td>
<td>1.81021</td>
<td>1.83762</td>
<td>2.248</td>
</tr>
<tr>
<td>8</td>
<td>0.9</td>
<td>0.9</td>
<td>0.93</td>
<td>0.81293</td>
<td>0.8142</td>
<td>1.3455</td>
</tr>
<tr>
<td>9</td>
<td>0.71</td>
<td>0.69</td>
<td>0.82</td>
<td>2.16665</td>
<td>2.20639</td>
<td>2.5805</td>
</tr>
<tr>
<td>10</td>
<td>0.77</td>
<td>0.75</td>
<td>0.85</td>
<td>1.22067</td>
<td>1.24205</td>
<td>1.7185</td>
</tr>
</tbody>
</table>

The following abbreviations are used in Table 1:

BTP = Estimated coverage probability of confidence interval based on bootstrap.

NAP = Estimated coverage probability of confidence interval based on normal approximation.

HTP = Estimated coverage probability of confidence interval based on normal approximation.
based on hypothesis testing.
BTL = Average length of confidence interval based on bootstrap.
NAL = Average length of confidence interval based on normal approximation.
HTL = Average length of confidence interval based on hypothesis testing.

6. Applications in Finance
The random variables \( R_1, R_2, \ldots, R_N \) in Section 1 may be considered to be the returns of \( N \) stocks, and the \( \gamma \)-quantile \( Q_\gamma \) of \( R \) becomes the value at risk (VaR) of the portfolio consisting of these \( N \) stocks. Thus, if we can show that \( R \) can be written as \( R = \mathbf{u} + \mathbf{AS} \) of which \( S_1, S_2, \ldots, S_N \) are uncorrelated and \( S_i \sim QN(0, \lambda_i) \), then the methods in Sections 2 and 4 can be applied to find confidence intervals for the VaR of the portfolio.

In the following analysis, the data obtained from the Kuala Lumpur Stock Exchange (KLSE) are used. The data are the daily stock prices of three companies, namely Genting Bhd., Gamuda Bhd. and Tanjong PLC Bhd. in the KLSE from Thomson Financial Datastream (01/01/1993 to 31/8/2002). The data for the period from 01/07/1997 to 30/06/1999 are excluded in the present investigation because these data were collected during the financial crisis in South East Asia. The following results in the forms of table and figure are extracted from Yap (2004).

Table 3. Variance-Covariance matrix associated with the portfolio

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4.6316</td>
<td>0.7453</td>
<td>1.2520</td>
</tr>
<tr>
<td>0.7453</td>
<td>4.0142</td>
<td>1.2299</td>
</tr>
<tr>
<td>1.2520</td>
<td>1.2299</td>
<td>5.7027</td>
</tr>
</tbody>
</table>

Table 4. The values of \( \lambda_{11}, \lambda_{12}, \lambda_{13} \) for the parameters of the fitted distributions for \( S_i, i = 1 \) to 3

<table>
<thead>
<tr>
<th>( \lambda_{11} )</th>
<th>( \lambda_{12} )</th>
<th>( \lambda_{13} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.174061</td>
<td>0.422125</td>
<td>-1.0</td>
</tr>
<tr>
<td>1.133999</td>
<td>0.553379</td>
<td>-1.0</td>
</tr>
<tr>
<td>1.541984</td>
<td>0.800266</td>
<td>-1.0</td>
</tr>
</tbody>
</table>

Figure 4 shows that the distribution of the portfolio returns \( R^{(P)} \) can be approximated well using the quadratic-normal distribution. Thus the methods in Sections 2 and 4 may be used to find confidence intervals for the VaR of the portfolio.

Figure 4.
Cumulative Distribution of return for the portfolio

References