BOND OPTION PRICING UNDER THE CKLS MODEL

C. Y. Khor[1], A. H. Pooi[2] and K. H. Ng[3]

[1]Faculty of Computing and Informatics, Multimedia University
Jalan Multimedia, 63100 Cyberjaya, Selangor, Malaysia.

[2]Sunway University Business School
No.5, Jalan Universiti, Bandar Sunway, 46150 Petaling Jaya, Selangor, Malaysia.

[3]Institute of Mathematical Sciences, Faculty of Science Building
University of Malaya, 50603 Kuala Lumpur, Malaysia.

e-mail: [1]cykhor@mmu.edu.my, [2]ahhinp@sunway.edu.my, [3]kokhaur@um.edu.my

Abstract**—***Consider the European call option written on a zero coupon bond. Suppose the call option has maturity T and strike price K while the bond has maturity $S > T$. We propose a numerical method for evaluating the call option price under the Chan, Karolyi, Longstaff and Sanders (CKLS) model in which the increment of the short rate over a time interval of length dt, apart from being independent and stationary, is having the quadratic-normal distribution with mean zero and variance dt. The key steps in the numerical procedure include (i) the discretization of the CKLS model; (ii) the quadratic approximation of the time-T bond price as a function of the short rate $r(T)$ at time T; and (iii) the application of recursive formulas to find the moments of $r(t+dt)$ given the value of $r(t)$. The numerical results thus found show that the option price decreases as the parameter γ in the CKLS model increases, and the variation of the option price is slight when the underlying distribution of the increment departs from the normal distribution.*

Keywords: *zero coupon bond, CKLS model, option price***.**

I. INTRODUCTION

One factor interest rate models are a well-known class of interest rate models in the pricing of interest rate derivatives. There are many types of one factor interest rate models, such as the Vasicek model, Cox, Ingersoll and Ross (CIR) model, Hull-White model, Chan, Karolyi, Longstaff and Sanders (CKLS) model and others (see Vasicek(1977)[1], CIR(1985)[2], Hull-white(1990)[3] and CKLS(1992)[4]). The Vasicek model is considered as the most fundamental one factor interest rate model which exhibits mean reversion. The Vasicek model follows the following stochastic differential equation:

$$dr(t) = a(m - r(t))dt + \sigma\, dw(t).$$

where $r(t)$ is the interest rate at time $t = 0$, *a* the drift factor with $a > 0$, m the long run mean around which mean reversion occurs, σ the volatility factor and $w(t)$ a Brownian motion. In mean reversion, if the interest rate $r(t)$ is above the long run mean *m*, then the coefficient of *a* performs a negative drift such that the interest rate is pulled down back to the long run mean *m*. Likewise, if the interest rate $r(t)$ is below the long run mean *m*, then the coefficient of *a* performs a positive drift such that the interest rate is pulled up back to the long run mean *m*. Thus, the coefficient of *a* is a speed of adjustment of interest rate towards *m* when it wanders away. However, there are some drawbacks of the Vasicek model. One of the main drawbacks is that the interest rate is theoretically possible to become negative which obviously does not make sense in market. This main shortcoming was then fixed in the CIR model. The CIR model follows the following stochastic differential equation:

$$dr(t) = a(m - r(t))dt + \sigma\sqrt{r(t)}\, dw(t).$$

In the CIR model, the term $\sqrt{r(t)}$ is imposed and its standard deviation factor becomes $\sigma\sqrt{r(t)}$. This standard deviation factor guarantees a nonnegative interest rate and hence eliminates the main drawback of the Vasicek model.

In 1992, Chan, Karolyi, Longstaff and Sanders introduced a model which covers both Vasicek and CIR models. The CKLS model follows the following stochastic differential equation:

$$dr(t) = a(m - r(t))dt + \sigma[r(t)]^{\gamma} dw(t); \quad r(0) = r_0, \qquad (1)$$

where *a* is the drift factor, *m* the mean around which mean reversion occurs, σ the volatility factor, γ a positive constant, r_0 the initial interest rate at $t = 0$ and $w(t)$ a Brownian motion. Analytical solutions have been found in many of the one factor interest rate models. Unfortunately, there are no analytical solutions for the bond and option prices based on the CKLS model. Therefore, numerical approaches are required. In 2008, Pooi et al. and Ng et al. as in [5] and [6] proposed numerical methods to approximate the distributions of the short term interest rates in one factor models. Barone-Adesi et al. (1997, 1999) as in [7] and [8] provided a method call the Box method for valuing the prices of the zero coupon bond and the call option written on the zero coupon bond in the CKLS model featured by a Brownian motion.

The discretized version of (1) featured by a Levy process with quadratic-normal increments can be expressed as

$$r_k = r_{k-1} + a(m - r_{k-1})\Delta t + \sigma r_{k-1}^{\gamma} w_k \sqrt{\Delta t},\ k \geq 1, \qquad (2)$$

where $r_k = r(t),\ t = k\Delta t,\ w_1, w_2, \cdots$ are independent, w_k has a quadratic-normal distribution (see Pooi (2003) [9]) with parameters 0 and $\bar{\lambda}$, [$w_k \sim QN(0,\bar{\lambda})$] and $\bar{\lambda}$ is such that $Var(w_k) = 1,\ k = 1,2,\cdots$.

Let B_T^S denotes the bond price at time $t = T$ of a zero coupon bond that matures at time $t = S$, $0 \leq T \leq S$. The bond price B_T^S can be expressed as

$$B_T^S = E\left(e^{-\int_T^S r(s)ds}\right). \qquad (3)$$

In 2009, Khor and Pooi as in [10] proposed a numerical method for evaluating the bond price under the CIR model featured by a Levy process. While in 2010, Khor, Ng and Pooi as in [11] proposed a numerical method for finding bond price of a zero coupon bond with maturity *T* under the CKLS model featured by a Levy process.

The price of the call option with maturity *T* and strike price *K* written on the zero coupon bond that matures at time *S* is given by

$$C = E\left\{\exp\left[-\int_0^T r(s)ds\right]\left[B_T^S - K\right]^+\right\}, \qquad (4)$$

where $\left[B_T^S - K\right]^+ = \max\left\{\left[B_T^S - K\right], 0\right\}$ is the payoff function.

In this paper, we present an alternative method based on numerical integration for finding the price of European call option written on a zero coupon bond when the interest rate follows the CKLS model featured by a Levy process.

The rest of this paper is structured as follows. In Section II, we explain two methods and an analytical formula of evaluating the price of the European call option written on a zero coupon bond. Section III shows some results found from the methods in Section II. The analysis of the results is discussed as well. We conclude our work in Section IV.

II. Methods of Evaluating the Price of the European Call Option Written on a Zero Coupon Bond

We present two methods of finding the price of the call option written on a zero coupon bond. These methods are respectively the simulation method and the method based on numerical integration.

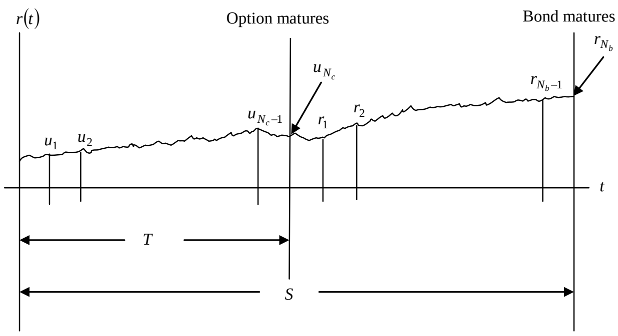

Figure 1. Short rate $r(t)$ and times of the maturity of Bond and Option

Suppose the interval $(0,T)$ is divided into N_c intervals each of length Δt and the interval (T,S) into N_b intervals each also of length Δt (see fig. 1). The two methods for finding the call option price written on a zero coupon bond are described below.

2.1 Simulation Method

We first use (2) to generate M_c values of $\mathbf{V}_c$ = $(u_1, u_2, \ldots, u_{N_c})\Delta t$ where u_i is the short rate at time $t = i\Delta t$. For each generated value of $\mathbf{V}_c$, we again use (2) to generate M_b values of $\mathbf{V}_b = (r_1, r_2, \ldots, r_{N_b})\Delta t$ where r_j is the short rate at time $t = T + j\Delta t$. Denote the i^{th} generated value of $\mathbf{V}_c$ by $\mathbf{V}_{c_i} = (u_{i1}, u_{i2}, \ldots, u_{iN_c})\Delta t$ and the j^{th} generated of $\mathbf{V}_b$ by $\mathbf{V}_{b_j} = (r_{j1}, r_{j2}, \ldots, r_{jN_b})\Delta t$. For the i^{th} generated value $\mathbf{V}_{c_i}$, we compute the following bond price, $p^{(i)}$ and payoff, $q^{(i)}$.

$$p^{(i)} = \frac{1}{M_b}\sum_{j=1}^{M_b} \exp\left(-V_{b_j}\right) \text{ and } q^{(i)} = \left\{p^{(i)} - K\right\}^+,$$

where $V_{b_j} = (r_{j1} + r_{j2} + \ldots + r_{jN_b})\Delta t$.

The price *C* of the European call option written on a zero coupon bond can be estimated using the following expression:

$$C = \frac{1}{M_c}\sum_{i=1}^{M_c} \exp\left(-V_{c_i}\right) q^{(i)},$$

where $V_{c_i} = (u_{i1} + u_{i2} + \ldots + u_{iN_c})\Delta t$.

2.2 Numerical Integration Method

In Khor, Ng and Pooi (2010) as in [11], the polynomial approximations of the first four moments of r_k (see (2)) conditioned on the given value of r_{k-1} are given as

$$E(r_k \mid r_{k-1}) = c_{10} + c_{11} r_{k-1}, \tag{5}$$

$$E(r_k^2 \mid r_{k-1}) = c_{20} + c_{21} r_{k-1} + c_{22} r_{k-1}^2, \tag{6}$$

$$E(r_k^3 \mid r_{k-1}) = c_{30} + c_{31} r_{k-1} + c_{32} r_{k-1}^2 + c_{33} r_{k-1}^3, \tag{7}$$

$$E(r_k^4 \mid r_{k-1}) = c_{40} + c_{41} r_{k-1} + c_{42} r_{k-1}^2 + c_{43} r_{k-1}^3 + c_{44} r_{k-1}^4, \tag{8}$$

where $c_{10}, c_{11}, \ldots, c_{44}$ are some constants.

We now use the polynomial approximations for the first four moments of r_k conditioned on the value of r_{k-1} to find the bond price B_T^S.

Let $V_b = (r_1 + r_2 + \ldots + r_{N_b})\Delta t$. Applying (5) with $k = N_b, N_b - 1, \ldots, 1$, we can get $E(V_b)$ in terms of a linear function of c_{10} and c_{11}. Applying (5) and (6) with $k = N_b, N_b - 1, \ldots, 1$, we can get $E(V_b^2)$ in terms of a quadratic function of c_{10}, c_{11}, c_{20}, c_{21} and c_{22}. Applying (5), (6) and (7), $E(V_b^3)$ is found in terms of a cubic function of c_{10}, c_{11}, c_{20}, c_{21}, c_{22}, c_{30}, c_{31}, c_{32} and c_{33}. Applying (5), (6), (7) and (8), we get $E(V_b^4)$ in terms of a quartic function of c_{10}, c_{11}, c_{20}, c_{21}, c_{22}, c_{30}, c_{31}, c_{32}, c_{33}, c_{40}, c_{41}, c_{42}, c_{43} and c_{44}.

By using (3), the following bond price can then can be obtained for a given value of r_0:

$$B_T^S \approx 1 - E(V_b) + \frac{E(V_b^2)}{2} - \frac{E(V_b^3)}{6} + \frac{E(V_b^4)}{24}.$$

After computing B_T^S for a number of selected values of r_0 in $(0,1)$, we then apply the regression technique to find the following quadratic approximation for B_T^S.

$$B_T^S \approx g_0 + g_1 r_0 + g_2 r_0^2 \tag{9}$$

We note that the value of r_0 coincides with the value u_{N_c} which appears in the vector $\mathbf{V}_{N_c} = (u_1, u_2, \ldots, u_{N_c})\Delta t$. Thus from (4) and (9), we get

$$C = E(\phi) \tag{10}$$

$$= E_{u_1|u_0} E_{u_2|u_1} E_{u_3|u_2} \ldots E_{u_{N_c}|u_{N_c-1}}(\phi),$$

where

$$\phi = \left\{ \exp\left[-\int_0^T r(s)ds \right] \left[\left(g_0 + g_1 u_{N_c} + g_2 u_{N_c}^2 \right) - K \right]^+ \right\}$$

$$= \left(1 - V_c + \frac{V_c^2}{2} - \frac{V_c^3}{6} + \frac{V_c^4}{24} \right) \left[\left(g_0 + g_1 u_{N_c} + g_2 u_{N_c}^2 \right) - K \right]^+,$$

with $V_c = (u_1 + u_2 + \ldots + u_{N_c})\Delta t$

Next, we let

$$F_{u_{N_c-1}} = E_{u_{N_c}|u_{N_c-1}}(\phi). \tag{11}$$

$$F_{u_{N_c-2}} = E_{u_{N_c-1}|u_{N_c-2}} E_{u_{N_c}|u_{N_c-1}}(\phi). \tag{12}$$

$$\vdots$$

$$F_{u_1} = E_{u_2|u_1} E_{u_3|u_2} \ldots E_{u_{N_c}|u_{N_c-1}}(\phi). \tag{13}$$

To find $F_{u_{N_c-1}}$, we first use numerical integration to compute

$$E_{\alpha, u_{N_c-1}} = E_{u_{N_c} | u_{N_c-1}} \left\{ u_{N_c}^{\alpha} \left[\left(g_0 + g_1 u_{N_c} + g_2 u_{N_c}^2 \right) - K \right] \right\},$$

$\alpha = 0,1,2,3,4,$ and express each $E_{\alpha, u_{N_c-1}}$ as a low degree polynomial function of u_{N_c-1}:

$$E_{\alpha, u_{N_c-1}} = \tau_{\alpha 0} + \tau_{\alpha 1} u_{N_c-1} + \tau_{\alpha 2} u_{N_c-1}^2, \quad \alpha = 0. \tag{14}$$

$$E_{\alpha, u_{N_c-1}} = \tau_{\alpha 0} + \tau_{\alpha 1} u_{N_c-1} + \tau_{\alpha 2} u_{N_c-1}^2 + \tau_{\alpha 3} u_{N_c-1}^3, \quad \alpha = 1,2. \tag{15}$$

$$E_{\alpha, u_{N_c-1}} = \tau_{\alpha 0} + \tau_{\alpha 1} u_{N_c-1} + \tau_{\alpha 2} u_{N_c-1}^2 + \tau_{\alpha 3} u_{N_c-1}^3 + \tau_{\alpha 4} u_{N_c-1}^4, \quad \alpha = 3,4. \tag{16}$$

By using (14), (15) and (16), we can find $F_{u_{N_c-1}}$ approximately (see (11)) and express $F_{u_{N_c-1}}$ as a low degree polynomial function $F_{u_{N_c-1}}$ of u_{N_c-1}.

We next use (5) – (8) to compute $F_{u_{N_c-2}}$ given by (12):

$$F_{u_{N_c-2}} = E_{u_{N_c-1}|u_{N_c-2}} F_{u_{N_c-1}}.$$

Similarly by using the iterative formulas in (5) – (8), we can also find $F_{u_{N_c-3}}, \ldots, F_{u_2}, F_{u_1}$ and *C*.

2.3 Analytical Formula

When the increment $w(t+dt) - w(t)$ in the CIR model is having the normal distribution with mean zero and variance *dt*, the analytical result for the price of the European call option written on a zero coupon bond was derived in [2]. The explicit expression for the price is given below.

$$ZBC(t,T,S,K) = P(t,S)\chi^2 \left(2\bar{r}[\rho + \varphi + B(T,S)]; \frac{4\kappa\theta}{\sigma^2}, \frac{2\rho^2 r(t) \exp\{h(T-t)\}}{\rho + \varphi + B(T,S)} \right) -$$

$$KP(t,T)\chi^2 \left(2\bar{r}[\rho + \varphi]; \frac{4\kappa\theta}{\sigma^2}, \frac{2\rho^2 r(t) \exp\{h(T-t)\}}{\rho + \varphi} \right),$$

where

$$P(t,U)=A(t,U)\exp[-B(t,U)r_0]\ ,\ \ U=T \text{ or } S\ ,$$

$$A(t,U)=\left(\frac{2h\exp\left[\frac{1}{2}(\kappa+h)(U-t)\right]}{2h+(\kappa+h)[\exp(h(U-t))-1]}\right)^{\frac{2\kappa\theta}{\sigma^2}},$$

$$B(t,U)=\frac{2[\exp((U-t)h)-1]}{2h+(\kappa+h)[\exp(h(U-t))-1]},$$

$$\bar{r}=\bar{r}(S-T)=\frac{\ln\left[A(T,S)/K\right]}{B(T,S)},$$

$$\rho=\rho(T-t)=\frac{2h}{\sigma^2[\exp\{h(T-t)\}-1]},$$

$$\varphi=\frac{\kappa+h}{\sigma^2},$$

$$h=\sqrt{\kappa^2+2\sigma^2}\ .$$

The function $\chi^2(x;\kappa,\lambda)$ is a non-central chi-squared cumulative distribution function with κ degree of freedom and non-centrality parameter λ . In Section III, the numerical results based on the above formula will be compared with those obtained by using the simulation method and the proposed numerical method.

III. Numerical Results

In this section, we present some numerical values for the prices of the European call option written on a zero coupon bond when the interest rate follows the CKLS model. We denote $E(w_k^3)$ and $E(w_k^4)$ as $\bar{m}_3$ and $\bar{m}_4$ respectively (see (2)). When the random variable w_k in (2) has a standard normal distribution, we have $\bar{m}_3=0$ and $\bar{m}_4=3.0$ .The following table shows the values of the parameters used.

TABLE 1. Values of Parameters

Parameter	Value
a	1.0
m	1.0
σ	1.0
Δt	$1/365$
r_0	0.1
K	0.4

Table 2 shows the option prices obtained by using the methods in Sections 2.1 to 2.3 when $\gamma=0.5$, $\bar{m}_3=0$ and $\bar{m}_4=3.0$

TABLE 2. Option Price when $\gamma=0.5$, $\bar{m}_3=0.0$ AND $\bar{m}_4=3.0$ $(T=k\Delta t)$

k	**Option Price**		
	Simulation	***Numerical Integration***	***Analytical Formula***
50	0.21827092935	0.2221749346	0.2202023774
100	0.1806177512	0.1863042828	0.1831610025
200	0.1272962859	0.1344418227	0.1307168901
300	0.0959408320	0.1003385409	0.0976330740
365	0.0809268886	0.0832110177	0.0822883281

This table shows that the option prices found from the methods of simulation and numerical integration (Section 2.1 and Section 2.2) agree fairly well with those found from the analytical method (Section 2.3).

When the random variable w_k is quadratic-normally distributed, the option prices obtained from the methods of simulation and numerical integration for the CIR model are shown in Table 3.

TABLE 3. Option Prices when $\gamma=0.5$ AND $T=365\Delta t$ FOR Various Values of $(\bar{m}_3,\bar{m}_4)$

$\bar{m}_3$	$\bar{m}_4$	**Option Price**	
		Simulation	***Numerical Integration***
0.0	2.6	0.0822254808	0.0832112631
0.0	8.0	0.0788519675	0.0832079504
0.5	6.2	0.0810694488	0.0830742762
3.0	20.0	0.0820817169	0.0823858031

From Table 3, we observe that when w_k has the quadratic-normal distribution, the option prices do not differ much from the result obtained by the analytical formula for the case when $\gamma=0.5$ and $(\bar{m}_3,\bar{m}_4)=(0,3)$.

At the same time, we produce option prices when the γ is varied from 0.1 to 0.9. Table 4 shows the option prices when $\bar{m}_3=0.0$ and $\bar{m}_4$ is varied whereas Table 5 gives the option prices when $\bar{m}_3$ is nonzero.

TABLE 4. Option Prices for Different Values of γ FOR $\bar{m}_3=0.0$ AND $T=365\Delta t$

γ	$\bar{m}_3=0.0$		
	$\bar{m}_4=2.6$	$\bar{m}_4=3.0$	$\bar{m}_4=8.0$
0.1	0.1003085479	0.100308491	0.100307757
0.2	0.0920748876	0.0920748095	0.092073829
0.3	0.0881931422	0.0881300059	0.088191230
0.4	0.0855136680	0.0855134764	0.0855110817
0.5	0.0832112631	0.0832110177	0.0832079504
0.6	0.0810360881	0.0810357640	0.0810317101
0.7	0.0789214506	0.0789209927	0.0789152552
0.8	0.0768586515	0.0768579490	0.0768491154
0.9	0.0748481852	0.0748470100	0.0748321504

TABLE 5. OPTION PRICES FOR DIFFERENT VALUES OF γ FOR $\overline{m}_3 \neq 0.0$ AND $T = 365\Delta t$

γ	$\overline{m}_3 = 0.5 \quad \overline{m}_4 = 6.2$	$\overline{m}_3 = 3.0 \quad \overline{m}_4 = 20.0$
0.1	0.1000572229	0.0987054007
0.2	0.0918911949	0.0909350768
0.3	0.0880423613	0.0872776490
0.4	0.0853754988	0.0846821386
0.5	0.0830742762	0.0823858031
0.6	0.0808936705	0.0801695653
0.7	0.0787690562	0.0779761557
0.8	0.0766911151	0.0757819327
0.9	0.0746554390	0.0735216583

From Tables 4 and 5, we see that the option prices decrease when γ increases, but the variation of $\overline{m}_3$ and $\overline{m}_4$ has not much influence on the prices.

IV. CONCLUDING REMARKS

As the parameter γ in the CKLS model has a significant effect on the option price, it is important that γ should be determined carefully. Although the values of $\overline{m}_3$ and $\overline{m}_4$ have only slight effects on the option price, it is also important to determine these values accurately. The reason is that when the total number of units of option traded is large, a small difference in option price might still have a significant effect on the total value of the large number of units of option.

ACKNOWLEDGMENT

We thank to the RCAEM-II 2012 for giving us a chance to participate in the conference.

REFERENCES

[1] Vasicek, O.A., "An equilibrium characterization of the term structure," Journal of Financial Economics, vol. 5, pp. 177-188, 1977.

[2] John C Cox, Jonathan E Ingersoll, and Stephen A Ross, "A theory of the term structure of interest rates," Econometrica, vol. 53, issue 2, March 1985.

[3] Hull, J., and A. White, "Pricing interest rate derivatives securities," The Review of Financial Studies, vol. 3, num. 4, pp. 573-592, 1990.

[4] Chan, K. C., G. A. Karolyi, F. A. Longstaff, and A. B. Sanders., "An empirical comparison of alternative models of the short-term interest rate," The Journal of Finance, vol. 47, no. 3, pp. 1209-1227, July 1992.

[5] A.H,Pooi, K.H,Ng and Y.C,Wong, "Distribution of short-term rate in one-factor models," Proceedings of 16th annual conference on Pacific Basin Finance Economics Accounting Management (PBFEAM2008), 2-4 July 2008.

[6] K.H,Ng, A.H,Pooi, Y.C,Wong and W.L,Beh, "Effects of non-normality on the distribution of short-term interest rate in the CIR model," Proceedings of 16th annual conference on Pacific Basin Finance Economics Accounting Management (PBFEAM2008), 2-4 July 2008.

[7] Barone-Adesi, G., Allergretto, W., Dinenis, W. and Sorwar G., "New Numerical Methods for the Evaluation of Interest Rate Contingent Claims," technical report, Center for Mathematical Trading and Finance, City University Business School, London, 1997.

[8] Barone-Adesi, G., Allergretto, W., Dinenis, W. and Sorwar G., "Valuation of Derivatives Based on CKLS Interest Rate Models," technical note no. 99-06, Center for Mathematical Trading and Finance, City University Business School, London, 1999.

[9] A. H. Pooi, "Effects of Non-Normality on Confidence Intervals in Linear Models," technical report no. 6/2003, Institute of Mathematical Sciences, University of Malaya, 2003.

[10] C.Y.Khor, A.H.Pooi, "A method for evaluating bond price under the CIR model," Proceedings of 5th Asian Mathematical Conference, vol. 1, pp. 182-187, June 2009.

[11] C.Y.Khor, K.H.Ng, A.H.Pooi, "Bond price under the CKLS model featured by a Levy process," conference paper of PBFEAM 2010, 23-25 July 2010.